Addition laws of failure probability and their applications in reliability analysis of structural system with multiple failure modes

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Abstract
Compared with the methods solving failure probability of the structural system with multiple failure modes, those solving failure probability of a single failure mode are simpler and more well-developed, thus in order to employ the latter to establish the former, the addition laws of the failure probability are derived mathematically by use of the basic principles of probability theory. In the derived addition laws, the failure probability of the structural system with n failure modes is expressed as a combination of the failure probabilities of 2^n/2 single failure modes. Therefore, the failure probability of the structural system with multiple failure modes can be solved by the well-developed methods for the failure probability of a single failure mode. After reviewing the boundary theories, such as the second-order boundary, the third-order boundary, and the linear programming based boundary for analyzing the failure probability of the structural system with multiple failure modes, the derived addition laws are applied to evaluate several former order joint failure probability involved in those boundary theories. Additionally, a new small-scale linear programming based boundary theory which can sufficiently reduce the scale of the linear programming model involved is proposed. Two numerical examples, including a series and a parallel structural system, are employed to demonstrate the accuracy and efficiency of proposed techniques.

Keywords
Addition laws of failure probability, separate failure probability, joint failure probability, total failure probability, boundary theory, linear programming

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Introduction
The structural system reliability analysis problem consists of many failure modes generally, which are correlated with each other and each of them is expressed by a limit state function (implicit or closed-form) commonly. Each limit state function demarcates the input variable space into safety and failure domains. An important task of structural system reliability analysis is to estimate the failure probability (FP) of the structural system with random input variables under these assumptions. For each single failure mode of the structural system, many available well-developed approaches can be employed for estimating its FP, such as the first-order second-moment method (FOSM), the importance sampling method (ISM), the line sampling method (LSM), etc. However, due to the dependence between these failure modes, it is difficult to apply these methods to the structural system with multiple failure modes directly. In response to this problem, Youn and Wang11 proposed the decomposition theorem which is called the addition law of the FP in this article, for the series structural systems. In this article this law is extended to both series and parallel structural systems and its generalized
form is developed. The addition law of the FP indicates that the FPs of the series and parallel systems with \( n \) failure modes can be expressed as the combination of the FPs of \( 2^n - 1 \) single failure modes. Theoretically, the addition laws can yield precise estimates of the FP of the structural system as long as the \( 2^n - 1 \) FPs of the single failure modes can be computed correctly. However, the derived addition laws are unsuitable for computing the FP of the structural system with a large number of failure modes, since the computational cost increases exponentially as the number of the failure modes. Fortunately, the addition laws of the FP are especially suitable for estimating the initial lower order joint FPs (JFPs) which are involved in the boundary theories.

Boundary theories\(^{12–15}\) such as the second-order boundary theory proposed by Ditlevsen,\(^{13}\) the third-order proposed by Zhang,\(^{15}\) and the linear programming (LP) based proposed by Song and Kiureghian\(^{16}\) are widely applied to the structural system reliability analysis problem. For the practical applications of these boundary theories, it is necessary to develop an algorithm for computing several initial lower JFP which is often a difficult work. As revealed by the above paragraph, the proposed addition laws can be used for this study with lower computational cost. In the three mentioned boundary theories, Ditlevsen and Zhang’s methods are applicable to the series systems, while Song’s method can be applied to general systems including the series and the parallel systems. One of the main disadvantages of Song’s method is that the scale of the LP involved increases exponentially with respect to the number of the failure modes of system.\(^{16}\) This shortcoming will be highlighted as the number of failure modes increases. In order to circumvent the difficulties of the LP, a new boundary theory based on small-scale LP model is proposed for the series and parallel systems. In this proposed method, the scale of the LP model increases linearly with respect to the number of the failure modes of system, thus the involved computational cost is lower.

The rest of this article is organized as follows. ‘Addition laws of FP’ section introduces the mathematical agreements as well as the addition laws of the FP for both the series and the parallel systems. ‘A review of the boundary theories’ section briefly reviews the past results of the boundary theory. ‘A new boundary theory based on small-scale LP’ section proposes a new boundary theory based on small-scale LP model. ‘Numerical examples’ section introduces two numerical examples including a series and a parallel system, respectively to demonstrate the addition laws of the FP and these boundary theories. The final section gives a conclusion to this article. The appendix lists the proof for the derived addition laws.

Addition laws of FP

Preparing work

Before proposing the addition laws of the FP, it is necessary to give some brief introduction on the mathematical agreements involved in this article.

Let \( g_i(x) \) denotes the limit state function of the \( i \)th failure mode, then the failure domain of this failure mode, denoted as \( F_i \), is defined as follows:

\[
F_i = \{ x : g_i(x) < 0 \} \tag{1}
\]

The FP of the \( i \)th failure mode, denoted as \( PF_i \) can be expressed as follows:

\[
PF_i = P(F_i) = P\{g_i(x) < 0\} \tag{2}
\]

Consider a series structural system with \( n \) failure modes, the failure of this structural system will happen if only one of these modes fails, thus the total failure domain of the series structural system, denoted as \( F_S^{(n)} \), is the union of the failure domains of all the \( n \) failure modes, i.e.

\[
F_S^{(n)} = \bigcup_{i=1}^{n} F_i \tag{3}
\]

Then the FP of this series structural system, denoted by \( P_S^{(n)} \), is the probability measure of the total failure domain \( F_S^{(n)} \), i.e.

\[
P_S^{(n)} = P\left( \bigcup_{i=1}^{n} F_i \right) \tag{4}
\]

Similarly, consider a parallel structural system with \( n \) failure modes, the failure of this structural system will happen only if all these \( n \) modes fail, thus the total failure domain \( F_P^{(n)} \) of the structural system is the intersection of the failure domains of all the \( n \) failure modes, i.e.

\[
F_P^{(n)} = \bigcap_{i=1}^{n} F_i \tag{5}
\]

Then the FP of this parallel system, denoted by \( P_P^{(n)} \), is the probability measure of \( F_P^{(n)} \), i.e.

\[
P_P^{(n)} = P\left( \bigcap_{i=1}^{n} F_i \right) \tag{6}
\]

For a series structural system, since the total failure domain of the system is the union of the failure
domains of all the failure modes involved, the total failure domain and the FP of this structural system will be expanded with the increased number of the failure modes. Similarly, the FP of a parallel structural system will be decreased with the increased number of the failure modes.

The $p$th order separate FP (SFP), denoted as $SFP_{i_1,i_2,...,i_p}$, is defined as follows

$$SFP_{i_1,i_2,...,i_p} = P\left\{ \bigcap_{j=1}^{p} g_{i_j} < 0 \right\} \tag{7}$$

where $i_j (j = 1, 2, \ldots, p, p \leq n)$ is an integer in the interval of $[1, n]$ and $i_j \neq i_k$. $\otimes$ denotes the multiplication operation $\times$ or division operation $\div$ and $\otimes_{j=1}^{p} g_{i_j}$ the multiplication or division operation of any $p$ limit state functions $g_{i_j}(x)$ ($i = 1, 2, \ldots, p$). Clearly, $SFP_{i_1,i_2,...,i_p}$ is the FP of a single failure mode with limit state function of

$$1 \otimes \left( \bigotimes_{j=1}^{p} g_{i_j} \right).$$

When $p = 2$ and $p = 3$, for example, the possible limit state functions corresponding to $SFP_{i_1,i_2,...,i_p}$ is presented in Table 1. Generally, there are $2^p$ possible limit state function corresponding to $SFP_{i_1,i_2,...,i_p}$.

When $p = 2$ and $p = 3$, the domains defined by

$$1 \otimes \left( \bigotimes_{j=1}^{p} g_{i_j} \right) < 0$$

are shown by the shadow in Figure 1. By the SFPs, we can transform the FP of the structural system with multiple failure modes into the combination of the FPs of the single failure modes which is the basic idea of this article. In this way one can solve the FP of a series and parallel system and the JFP involved in the boundary theory\textsuperscript{12–16} by solving the SFPs since they are all the linear combination of the SFPs, as shown in the addition laws of FP proposed in the next subsection.

The $p$th order JFP, denoted by $JFP_{i_1,i_2,...,i_p}$, is defined as

$$JFP_{i_1,i_2,...,i_p} = P\left( \bigcup_{j=1}^{n} F_{i_j} \right) \tag{8}$$

This definition shows that $JFP_{i_1,i_2,...,i_p}$ is the FP of a parallel system with $p$ failure modes.

The $p$th order Total FP (TFP), denoted by $TFP_{i_1,i_2,...,i_p}$, is defined as

$$TFP_{i_1,i_2,...,i_p} = P\left( \bigcup_{j=1}^{n} F_{i_j} \right) \tag{9}$$

This definition shows that $TFP_{i_1,i_2,...,i_p}$ is the FP of a series system with $p$ failure modes.

The definitions of SFP, JFP and TFP show that

$$SFP_i = JFP_i = TFP_i = P F_i \quad i = 1, 2, \ldots, n \tag{10}$$

After preparing these works, the addition laws of the FP for the series and the parallel systems are carried out in the following subsection.

### Addition laws of the FP

The addition laws of the FP for the series and the parallel systems with $n$ failure modes are given by equations (11) and (12), respectively

$$P_s^{(n)} = P\left( \bigcup_{i=1}^{n} F_i \right) = \frac{1}{2^n-1} \left\{ \sum_{i=1}^{n} SFP_i + \sum_{i \neq j} SFP_{i,j} + \sum_{i \neq j \neq k} SFP_{i,j,k} + \cdots + SFP_{1,2,\ldots,n} \right\} \tag{11}$$

### Table 1. The possible limit state functions corresponding to $SFP_{1,2,...,p}$ when $p = 2$ and $p = 3$.

<table>
<thead>
<tr>
<th>Possible limit state functions</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times g_0 \times g_3$, $(1 \times g_0) \div g_0$, $(1 \div g_0) \times g_0$, $1 \div g_0 \div g_3$</td>
<td>$1 \times g_0 \times g_3$, $(1 \div g_0) \times g_0$, $(1 \div g_0) \div g_3$, $(1 \times g_0) \times g_0$, $(1 \div g_0) \div g_3$</td>
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SFP: separate failure probability.
\[ P_{P}^{(n)} = P\left(\bigcap_{i=1}^{n} F_{i}\right) = \frac{1}{2^{n-1}} \left\{ \sum_{i=1}^{n} SFP_{i} - \sum_{i \neq j}^{n} SFP_{i,j} \right\} + \sum_{i \neq j \neq k}^{n} SFP_{i,j,k} - \cdots + (-1)^{n-1} SFP_{1,2,\ldots,n} \] (12)

A simpler version of equation (11) is first proposed in Youn and Wang. In this article, we offer an alternative way to prove these laws and the proof is listed in the appendix. These addition laws indicate that the FP of both the series and parallel systems, defined in equations (4) and (6), respectively, can be expressed as a combination of the \( p \)th order SFPs, where \( p = 1, 2, \ldots, n \).

As mentioned in ‘Preparing work’ section, the \( p \)th order SFP is actually the FP of the structural system. In the boundary theories, this direct way of using the addition laws to compute the FP of the structural system is simple and can yield an accurate result if each SFP is evaluated completely. But for the system with a large number of failure modes, there are too many SFPs required to be evaluated which is often computationally expensive. What is more, the limit state function involved in the high order SFPs may be so complicated that many methods such as FOSM cannot work well. This leads us into the realm of boundary theories for solving the FP of the structural system. In the boundary theories, several initial lower order JFPs involved can be conveniently evaluated by the addition laws. Thus, the boundary theories are reviewed in the next section.

A review of the boundary theories

The boundary theories are a group of procedures for computing the FP of the structural system with the information of several lower order JFPs. In these approaches, the second-order boundary theory developed by Ditlevsen, the third-order developed by Zhang, and the LP based developed by Song and Kuo, widely used in the past years, are first reviewed. And the improved boundary theory is proposed using the addition laws of the FP in this section.

The second- and third-order boundary theories

Due to the addition laws of probability in the probability theory, the FP of a series system with \( n \) failure modes can be given as follows

\[ P_{S}^{(n)} = \sum_{i=1}^{n} JFP_{i} - \sum_{i \neq j}^{n} JFP_{i,j} + \sum_{i \neq j \neq k}^{n} JFP_{i,j,k} - \cdots + (-1)^{n-1} JFP_{1,2,\ldots,n} \] (14)

Obviously, the first-order probabilities of equation (14), i.e. \( \sum_{i=1}^{n} JFP_{i} \) can be regarded as an upper bound of \( P_{S}^{(n)} \) and the initial two lower order probabilities, i.e. \( \sum_{i=1}^{n} JFP_{i} - \sum_{i \neq j}^{n} JFP_{i,j} \) can be regarded as an lower bound. Again, the initial three and four lower probabilities can be regarded as upper and lower bounds, respectively. This process is repeated till all the \( n \) orders are considered, then the exact value of \( P_{S}^{(n)} \) can be obtained.

In equation (14), as the initial two lower order probabilities are considered, a narrow bound was proposed by Ditlevsen to be

\[ P_{S}^{(n)} \geq JFP_{1} + \sum_{i=2}^{n} \max \left\{ JFP_{1} - \sum_{j=1}^{i-1} JFP_{j,i,j} \right\} \] (15)

\[ P_{S}^{(n)} \leq \sum_{i=1}^{n} JFP_{i} - \sum_{i=2}^{n} \max \left\{ JFP_{i,j} \right\} \] (16)

To evaluate \( P_{S}^{(n)} \) by equations (15) and (16), \( JFP_{i,j} \) must be computed first. Suppose the limit state functions \( g_{i} \) and \( g_{j} \) are linear with independently standard normal input variables \( x_{k} \) (\( k = 1, 2, \ldots, m \)), i.e.

\[ g_{i}(x) = a_{0} + \sum_{k=1}^{m} a_{k} x_{k}, \quad g_{j}(x) = b_{0} + \sum_{k=1}^{m} b_{k} x_{k} \] (17)

where \( a_{i} \) and \( b_{i} \) (\( i = 0, 1, \ldots, m \)) are constants.

Then \( JFP_{i,j} \) can be evaluated with a narrow bound which is proposed by Ditlevsen as follows

\[ JFP_{i,j} \leq \Phi(-\beta_{i})\Phi\left(-\frac{\beta_{i} - \rho_{ij}}{\sqrt{1-\rho^{2}}}\right) + \Phi(-\beta_{j})\Phi\left(-\frac{\beta_{j} - \rho_{ij}}{\sqrt{1-\rho^{2}}}\right) \]

\[ JFP_{i,j} \geq \max \left\{ \Phi(-\beta_{i})\Phi\left(-\frac{\beta_{i} - \rho_{ij}}{\sqrt{1-\rho^{2}}}\right), 0 < \rho < 1 \right\} \]

\[ \Phi(-\beta_{i})\Phi\left(-\frac{\beta_{i} - \rho_{ij}}{\sqrt{1-\rho^{2}}}\right) \]

(18)
where $\beta_i$ and $\beta_j$ are the reliability indices of the $i$th and $j$th failure modes, respectively, $\Phi(\cdot)$ the cumulative distribution function of standard normal random variable, and $\rho$ the associated correlation coefficient of the $i$th and $j$th failure modes which is expressed as follows

$$\rho = \frac{\sum_{k=1}^{n} a_k b_k}{\left(\sum_{k=1}^{m} a_k^2\right)^{1/2} \left(\sum_{k=1}^{m} b_k^2\right)^{1/2}}$$

(20)

Considered the initial three order of equation (14), a third-order narrow bound proposed by Zhang is given as follows

$$P_{SP}^{(3)} \geq JP_F + JP_F - JP_{1,2}$$

$$+ \sum_{i=3}^{n} \max_{j \leq i} \left(0, JP_F - \sum_{j=1}^{i-1} JP_{i,j} + \max_{k \in \{1, \ldots, i-1\}} \sum_{j=1}^{i-1} JP_{i,j,k}\right)$$

(21)

$$P_{SP}^{(3)} \leq JP_F + JP_F - JP_{1,2}$$

$$+ \sum_{i=3}^{n} \left(JP_F - \max_{j < k, j \leq i-1} \left(JP_{i,k} + JP_{i,j} - JP_{i,j,k}\right)\right)$$

(22)

It should be noted that these second and third-order bounds are only applicable to the series system. For the parallel and general system, a relatively plausible procedure is the LP-based bounds developed by Song and Kiureghian, and it is reviewed in the next subsection.

**The LP-based bounds**

In this procedure, the sample space of the input variables of a system with $n$ failure modes is divided into $2^n$ mutually exclusive and collectively exhaustive areas, each of which consists of a distinct intersection of the failure domain $F_i$ and their complements $\bar{F}_i$. Take a system with three failure modes as an example, the $2^3$ domains are illustrated in Figure 2. Let $p_i (i = 1, 2, \ldots, 8)$ denote the probability measure of the domain $f_i$, then we have

$$\sum_{i=1}^{8} p_i = 1$$

(23)

$$p_i \geq 0, \, i = 1, 2, \ldots, 8$$

(24)

$$P_{SP}^{(3)} = c_1^T p$$

(25)

$$P_{SP}^{(3)} = c_1^T p$$

(26)

where $c_1 = (11111110)^T$, $c_2 = (1000000000)^T$, and $p = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)^T$. $c_1^T$ and $c_2^T$ are the transpositions of $c_1$ and $c_2$, respectively.

Equations (25) and (26) indicate that the FPs of the series and parallel systems are both the linear combinations of $p_i$. In fact, the FPs of any general system are all the linear combinations of $p_i$, such as

$$P(F_1, F_2, F_3) = c_3^T p$$

(27)

and

$$P((F_1 \cup F_2)(F_2 \cup F_3)) = c_4^T p$$

(28)

where $c_3 = (111110000)^T$ and $c_4 = (11100010)^T$.

Obviously, $p_i (i = 1, 2, \ldots, 8)$ should satisfy the following constraints while the first and second-order FPs are considered.

$$JP_{1,2} = p_1 + p_3 + p_4 + p_7$$

(29)

$$JP_{2,3} = p_1 + p_2 + p_4 + p_6$$

(30)

Equations (29) and (30) indicate that the first and second-order FPs are the linear combinations of $p_i$ which is also true for more general case.

Above all, the LP model for solving the FP of the structural system by Song and Kiureghian is given below

$$\min (\max) \quad c^T p$$

s. t. \quad \begin{align*}
    a p &= b \\
    p_i &\geq 0 (i = 1, 2, \ldots, 8^n)
\end{align*}$$

(31)

where $p$ is the design variable vector, $c^T$ the coefficient vector of the objective function which equals $c_1^T$ for the
series system and $c^T$ for the parallel system, $a$ the coefficient matrix of the equality constraints which can be obtained from equations (29) and (30), and $b = (1, JFP_1, JFP_2, JFP_3, JFP_{1,2}, JFP_{1,3}, JFP_{2,3})^T$.

Suppose the first and second-order JFPs have been computed beforehand, then the upper and lower bounds of the FPs of any systems can be evaluated by the LP model above. Regarding equations, (23), (29), and (30) as the equality constraints, equation (24) the inequality constraint, and equation (25) the objective function, then the upper and lower bounds of the FP of the series system can be evaluated by LP-based model. If the objective function is exchanged to equation (26), (27), or (28), then the upper and lower bounds of the FP of the parallel system or the general systems corresponding to equations (27) and (28) can be evaluated.

**Discussion of the boundary theories and their improvements by the derived addition laws**

Ditlevsen’s second-order boundary theory has been widely used for the reliability analysis of the series structural systems in the past few decades. However, two main disadvantages exist in this method. First, the bounds of the JFP illustrated in equations (18) and (19) are quite wide if the associated constant $\rho$ is too large. Second, equations (18) and (19) are the bounds of the JFP for the case that all the limit state functions are linear. For the series system with nonlinear limit state functions, there is no definite expression for the bounds of the JFP. A simple way to overcome these two disadvantages is to compute the JFP by the addition law of the FP instead of equations (18) and (19). Due to the addition law of the parallel system

$$JFP_{i,j} = \frac{1}{2} \{SFP_i + SFP_j - SFP_{i,j}\}$$

where $SFP_i$, $SFP_j$, and $SFP_{i,j}$ can be evaluated by FOSM, ISM or other methods for reliability analysis of a single failure mode. In this way, the evaluation of the associated constant $\rho$ is avoided and for the systems with nonlinear limit state functions, more accurate estimates for the second-order JFP can be obtained.

To compare the accuracy of equations (18) and (19) with the derived addition laws, we consider a series structural system with two failure modes. The limit state functions are given as follows.

$$g_1(x) = 2x_1 + x_2 + x_3 - 3x_4,$$
$$g_2(x) = x_1 + x_2 + 2x_3 - 3x_4$$

Four input variables $x_i$ ($i = 1, \ldots, 4$) are all normal variables with mean values $\mu_i = 1$ and SDs $\sigma_i = 0.08$. Then the reliability indexes of $g_1(x)$ and $g_2(x)$ are $\beta_1 = \beta_2 = 3.2275$, respectively and the associated correlation coefficient of these two failure modes is $\rho_{1,2} = 0.9333$. Then the bounds of $JFP_{1,2}$ computed.
by equation (18) are \([1.7135, 3.4270] \times 10^{-4}\). Computing the first and second-order SFPs by FOSM and substituting the estimates into equation (32), then we yield \(JFP_{1,2} = 3.3284 \times 10^{-4}\). The reference results of \(JFP_{1,2}\) (by Monte Carlo simulation (MCS)) is \(3.2460 \times 10^{-4}\). Apparently, the addition laws provide a much better estimate for \(JFP_{1,2}\) than Ditlevsen’s boundary theory. While the limit state functions are implicit, for estimating \(JFP_{i,j}\) by Ditlevsen’s boundary theory, one should first expend the limit state functions at the mean points or at the most probable points by Taylor series and obtain the linear approximation of the limit state functions. Then the reliability indices and the correlation coefficient of each failure mode can be computed. If the linearly closed-form limit state functions of each failure mode have been obtained, one can compute \(SFP_i, SFP_j, \text{ and } SFP_{i,j}\) by simulation method or FOSM method without evaluating the implicit limit state functions. Thus, the computational costs of Ditlevsen’s method and the addition laws are nearly the same.

Zhang\(^1\) introduced several integrals to calculate the second and third-order JFPs involved in equations (21) and (22). However, the integrand often may not be expressed analytically, thus it is difficult to be evaluated. Similarly, we can employ the addition law of the FP to evaluate the second and third-order JFPs instead of the integrals. The second-order JFP can be evaluated by equation (32). And the third-order JFP can be evaluated by the addition law of the parallel system as follows

\[
JFP_{i,j,k} = \frac{1}{4} \left\{ SFP_i + SFP_j + SFP_k - SFP_{i,j} ight. \\
\left. - SFP_{i,k} - SFP_{j,k} + SFP_{i,j,k} \right\} 
\]

Each of the SFPs in equation (34) can be evaluated by FOSM, ISM or other methods for reliability analysis with a single failure mode.

Song’s boundary theory also involves the evaluations of the JFPs which now can also be performed by equations (32) and (34). Song’s boundary theory has several definite advantages. First, as mentioned earlier, it is applicable to all systems including series, parallel, and general systems. Second, Song’s boundary theory can provide the narrowest bound with the information of the JFPs,\(^1\)\(^6\) which cannot be reached by the Ditlevsen and Zhang’s methods. Third, it is very convenient to be extended to higher order bounds if the information of higher order JFPs can be obtained. A fatal drawback of Song’s boundary theory is that the scale of LP model involved increases exponentially with respect to the number of the failure modes. For the system with \(n\) failure modes, the number of the design variables is \(2^n\), if the initial three-order JFPs are considered as the constraints, then the number of the equality constraints is

\[
1 + C_n^1 + C_n^2 + C_n^3 = \frac{n^3 + 5n + 6}{6} 
\]

Thus, the computational cost involved in Song’s boundary theory will be tremendously heavy if the number of failure modes is too large. Another drawback is that it is a hard work to carry out a general program for Song’s boundary theory. In order to overcome these two disadvantages, a new boundary theory involving small-scale LP model is proposed in next section.

### A new boundary theory based on small-scale LP

The expensive computational cost involved in Song’s boundary theory results from the division of the sample space of the input variables. In this section, we divide the sample space \(S\) into \(n + 1\) mutually exclusive and collectively exhaustive domains which is shown as follows

\[
S = \bigcup_{i=0}^{n} A_i
\]

where \(A_i (i = 0, 1, \ldots, n)\) denotes the area where \(i\) modes fail and \(A_i \cap A_j = \emptyset\) for \(i \neq j\). Let \(P_i\) denote the probability measure of \(A_i\). When \(n = 2\) and \(n = 3\), the division of the sample space is schematically shown in Figure 3.

Clearly, the following equations hold

\[
P^{(n)}_S = E_S^T P
\]

\[
P^{(n)}_P = E_P^T P
\]

where \(E_S = (0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T\), \(E_P = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T\), both of which are \(n + 1\) dimensional vector, and \(P = (P_0 \ P_1 \ldots \ P_n)^T\). equations (37) and (38) show that both the FPs of series and parallel system can be represented as the linear combinations of \(P_i\), thus they can be regarded as the objective functions of a LP model.

Next we will show that the JFPs can also be represented by the linearly combination of \(P_i\), thus can be regarded as the equality constraints of a LP model. From equation (88) (given in appendix), equations (39) to (41) can be obtained

\[
\sum_{i=1}^{n} JFP_i = P_1 + 2P_2 + 3P_3 + \cdots + nP_n
\]
Equations (39) to (41) show that the summations of JFPs can be represented by the linearly combinations of $P_i$, thus they can be regarded as the equality constraints of a LP model. In fact, the information of TFPs and that of SFPs can also be regarded as the equality constraints as long as the TFPs and SFPs can be represented by a linearly combinations of $P_i$. From the addition law of probability in probability theory we know that

$$TFP_{i,j} = JFP_i + JFP_j - JFP_{i,j}$$

then

$$\sum_{i \neq j} TFP_{i,j} = \sum_{i \neq j} (JFP_i + JFP_j - JFP_{i,j})
= (n-1) \sum_{i=1}^{n} JFP_i - \sum_{i \neq j} JFP_{i,j}$$

(43)

Substituting equation (40) into equation (43) we obtain

$$\sum_{i \neq j} TFP_{i,j} = \sum_{i=1}^{n} i \left( n - 1 - \frac{i-1}{2} \right) P_i$$

(44)

Similarly, we have

$$\sum_{i \neq j \neq k} TFP_{i,j,k} = \sum_{i=1}^{n} i \left( \frac{(n-1)(n-2)}{2} - \frac{(i-1)(n-2)}{2}
+ \frac{(i-1)(i-2)}{6} \right) P_i$$

(45)

From the addition laws of the FP for the series and parallel systems we have

$$\sum_{i \neq j} JFP_{i,j} = \frac{1}{2} \sum_{i \neq j} (SFP_i + SFP_j - SFP_{i,j})$$

(46)

$$\sum_{i \neq j} TFP_{i,j} = \frac{1}{2} \sum_{i \neq j} (SFP_i + SFP_j + SFP_{i,j})$$

(47)

Subtracting equation (46) from equation (47) yields

$$\sum_{i \neq j} SFP_{i,j} = \sum_{i \neq j} TFP_{i,j} - \sum_{i \neq j} JFP_{i,j}$$

(48)

Substituting equations (40) and (44) into equation (48) yields

$$\sum_{i \neq j} SFP_{i,j} = \sum_{i=1}^{n} i(n-i)P_i$$

(49)

Similarly we can obtain

$$\sum_{i \neq j \neq k} SFP_{i,j,k} = \sum_{i=1}^{n} i \left( \frac{(n-1)(n-2)}{2} - \frac{(i-1)(n-2)}{2}
+ \frac{2}{3}(i-1)(i-2) \right) P_i$$

(50)

Note that

$$P_0 + P_1 + \cdots + P_n = 1$$

(51)

Then the following LP model can be developed to evaluate the bounds of the structural system FP while the initial three-order JFPs or TFPs or SFPs are considered.

$$\max \ (\min) \ E^TP (E^TP)$$

s. t. $EP = B$

$P_i \geq 0 \ (i = 0, 1, \ldots, n)$

(52)
where \( P \) is the design variable vector and \( E \) the coefficient matrix of the equality constraints. If the information of the initial three-order JFPs are considered as equality constraints, then

\[
E = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & 3 & 4 & \cdots & n(n-1) \\
0 & 0 & 1 & 3 & 6 & \cdots & \frac{n(n-1)(n-2)}{2} \\
0 & 0 & 0 & 1 & 4 & \cdots & \frac{n(n-1)(n-2)}{6}
\end{bmatrix}
\]  
(53)

\[
B = \left( 1 \sum_{i=1}^{n} \sum_{j \neq j} JFP_{i, j} \sum_{i \neq j \neq k} JFP_{i, j, k} \right)^T
\]  
(54)

If the information of the initial three-order TFPs are considered as equality constraints, then

\[
E = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & n \\
0 & n-1 & 2n-3 & \cdots & n(n-1) \\
0 & \frac{(n-1)(n-2)}{2} & (n-2)(n-3) & \cdots & \frac{n(n-1)(n-2)}{6}
\end{bmatrix}
\]  
(55)

\[
B = \left( 1 \sum_{i=1}^{n} \sum_{j \neq j} TFP_{i, j} \sum_{i \neq j \neq k} TFP_{i, j, k} \right)^T
\]  
(56)

If the information of the initial three SFPs are considered as equality constraints, then

\[
E = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & n \\
0 & n-1 & 2n-3 & \cdots & 0 \\
0 & \frac{(n-1)(n-2)}{2} & (n-2)(n-3) & \cdots & 0
\end{bmatrix}
\]  
(57)

\[
B = \left( 1 \sum_{i=1}^{n} \sum_{j \neq j} SFP_{i, j} \sum_{i \neq j \neq k} SFP_{i, j, k} \right)^T
\]  
(58)

Theoretically, this approach is applicable to the series and parallel systems with a large number of failure modes. However, as will be demonstrated in the second numerical example, when this small-scale LP-based boundary theory is employed to the parallel system, the estimates of the FP of the system often turn out to be unpractical, i.e. the lower bound is too small to be accepted. In the content of structural system reliability analysis, one often pays more attention to the upper bound other than the lower bound, thus this small-scale LP-based bound is valuable to parallel system despite the not precise enough estimates. Here we introduce another upper bound for the FP of parallel system.

Apparently, the equations (59) and (60) hold

\[
F_i = \left( F_i \cap F_j \right) \cup \left( F_i \cap F_j \right)
\]  
(59)

\[
\left( F_i \cap F_j \right) \cap \left( F_i \cap F_j \right) = \emptyset
\]  
(60)

Thus

\[
JFP_i = JFP_{i, j} + P(F_i \cap F_j)
\]  
(61)

Then we can obtain equations (62) to (64)

\[
\min JFP_{i, j} \leq JFP_i
\]  
(62)

\[
\min JFP_{i, j} \leq \min_{i=1}^{n} JFP_i
\]  
(63)

\[
JFP_{1, 2, \ldots, n} \leq \cdots \leq \min_{i=1}^{n} JFP_{i, j} \leq \min_{i=1}^{n} JFP_i
\]  
(64)

Note \( JFP_{1, 2, \ldots, n} = P_p^{(n)} \), while the former two-order JFPs are considered, \( \min_{i \neq j} JFP_{i, j} \) can be regarded as a upper bound of \( P_p^{(n)} \), while the initial three-order JFPs are considered, \( \min_{i \neq j \neq k} JFP_{i, j, k} \) can be regarded as a upper bound of \( P_p^{(n)} \). The higher order JFPs are considered, the lower estimate of the upper bound of \( P_p^{(n)} \) can be obtained. We denote this upper bound as \( B_{(n)}^{(2)} \) when considering the initial two-order JFPs and \( B_{(n)}^{(3)} \) when considering the initial three-order JFPs. It should be noted that the second-order upper bound \( B_{(n)}^{(2)} \) is first proposed by Murotsu et al. \( \text{17} \) and \( B_{(n)}^{(3)} \) can be estimated by the addition laws of the FP.

### Numerical examples

In this section, two simple numerical examples are employed to demonstrate the addition laws of the FP and the several boundary theories. Without loss of generality, it is assumed that the limit state function corresponding to each SFP equals the product of several limit state functions corresponding to the \( n \) failure.
modes and there is no division operation involved. We also just consider the information of the JFPs as the constraints when implement the small-scale LP model based boundary theory.

**Example 1: Plain truss structure**

Consider a plain truss structure shown in Figure 4 and five potential failure modes of the system can be readily identified and defined by the following linear limit state functions\(^{18}\)

\[
\begin{align*}
g_1 &= R_2 + \sqrt{2}R_3/2 - T_2 \\
g_2 &= R_1 + \sqrt{2}R_3/2 - T_2 - T_1 \\
g_3 &= 2R_1 - T_2 \\
g_4 &= \sqrt{2}R_3 - T_1 \\
g_5 &= R_1 + R_2 - T_1 + T_2
\end{align*}
\]

The limit state function \(g\) of this series system is given by 
\(g = \min\{g_i\}\). \(R_i\) (\(i = 1, 2, 3\)) are the compressive or tensile strength of the three trusses, \(T_1\) and \(T_2\) are the point loads. The five input variables, i.e. \(R_i\) (\(i = 1, 2, 3\)), \(T_1\) and \(T_2\) are independently normal random variables whose distribution parameters are presented in Table 2.

![Figure 4. A plain truss structure.](image)

**Table 2.** Distribution parameters of the input variables of example 1.

<table>
<thead>
<tr>
<th>Input variable</th>
<th>(R_1)</th>
<th>(R_2)</th>
<th>(R_3)</th>
<th>(T_1)</th>
<th>(T_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean value (kN)</td>
<td>137.2</td>
<td>98.0</td>
<td>196.0</td>
<td>29.4</td>
<td>127.4</td>
</tr>
<tr>
<td>Coefficient of variation</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The results of the SFPs evaluated by FOSM, ISM, and MCS are presented in Table 3, where \(N\) is the total number of the limit state function evaluations and \(N\) can be used to measure the computational efficiency of these three methods. The numbers of simulations of ISM and MCS are 1000 and \(10^7\), respectively. Due to the addition law of series system

\[
P_S^{(5)} = \frac{1}{16} \left\{ \sum_{i=1}^{5} SFP_i + \sum_{i \neq j} SFP_{i,j} + \sum_{i \neq j \neq k} SFP_{i,j,k} + \sum_{i \neq j \neq k \neq m} SFP_{i,j,k,m} + SFP_{1,2,3,4,5} \right\}
\]

Substituting the results of SFP into equation (65), the FPs of the system by FOSM, ISM, and MCS are presented in the last column of Table 3.

Results estimated by FOSM in Table 3 show that this method provide a good estimates for the low order SFPs, but not for the high order SFPs such as \(SFP_{1,2,3,4,5}\). This is caused by the complex character of the limit state function \(\prod_{i=1}^{5} g_i\). The advantage of FOSM to evaluate the SFP is that the total number of function evaluations is 528 as opposed to 31582 and \(5 \times 10^7\) required for ISM and MCS, respectively. This clearly shows the computational efficiency of FOSM. It is also presented in Table 3 that results of both low and high order SFPs by ISM are accurate enough compared with that by MCS, but efficiency of ISM is lower than FOSM. This leads us to an eclectic idea of computing the low order SFPs by FOSM and high order SFPs by ISM. The result of the FP of this series system by direct MCS with \(10^7\) simulations is \(2.848 \times 10^{-4}\) which approximately equals the result by MCS based on equation (65), i.e. \(2.939 \times 10^{-4}\). This fact verifies the correctness of the addition law of the FP. The results of the FP of the system by FOSM and ISM based on equation (65), presented in the last column of Table 3 are \(2.342 \times 10^{-4}\) and \(2.868 \times 10^{-4}\), respectively.

Substituting the results of the initial three-order SFPs evaluated by MCS presented in Table 3 into equations (32) and (34), the estimates of the former three-order JFPs can be obtained, which are presented...
in Table 4. By substituting these results into equations, (15), (16), (21), and (22), we can obtain the Ditlevsen and Zhang's bounds presented in the second and third row of Table 5, respectively. Song's bounds and small-scale LP-based bounds can also be computed by solving corresponding LP model with constraints of the initial three-order JFP and the results are also presented in Table 5. It is shown that all these bounds are accurate as they are expected.

**Example 2: A numerical example**
Consider a parallel system with five failure modes, the limit state functions are given as follows

\[
\begin{align*}
    g_1(x) &= \sqrt{2}x_1 + x_2 - x_5 \\
    g_2(x) &= x_1 + x_3 - x_5 \\
    g_3(x) &= x_2/\sqrt{2} + x_3 - x_5 \\
    g_4(x) &= x_2 + x_4/\sqrt{2} - x_5 \\
    g_5(x) &= x_3 + x_4/\sqrt{2} - x_5
\end{align*}
\]

**Table 3.** Results of the SFP's estimated by FOSM, ISM, and MCS ($\times 10^{-4}$).

<table>
<thead>
<tr>
<th>SFP</th>
<th>FOSM</th>
<th>ISM</th>
<th>MCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SFP_1</td>
<td>1.809</td>
<td>1.794</td>
<td>1.837</td>
</tr>
<tr>
<td>SFP_2</td>
<td>1.108</td>
<td>1.087</td>
<td>1.132</td>
</tr>
<tr>
<td>SFP_3</td>
<td>0.432</td>
<td>0.433</td>
<td>0.432</td>
</tr>
<tr>
<td>SFP_4</td>
<td>0.347</td>
<td>0.349</td>
<td>0.359</td>
</tr>
<tr>
<td>SFP_5</td>
<td>0.039</td>
<td>0.040</td>
<td>0.031</td>
</tr>
<tr>
<td>SFP_1,2</td>
<td>1.809</td>
<td>1.998</td>
<td>2.217</td>
</tr>
<tr>
<td>SFP_1,3</td>
<td>1.809</td>
<td>1.809</td>
<td>2.072</td>
</tr>
<tr>
<td>SFP_1,4</td>
<td>1.809</td>
<td>1.673</td>
<td>1.720</td>
</tr>
<tr>
<td>SFP_1,5</td>
<td>1.809</td>
<td>1.774</td>
<td>1.830</td>
</tr>
<tr>
<td>SFP_2,3</td>
<td>1.108</td>
<td>1.177</td>
<td>1.183</td>
</tr>
<tr>
<td>SFP_2,4</td>
<td>1.089</td>
<td>2.154</td>
<td>2.226</td>
</tr>
<tr>
<td>SFP_2,5</td>
<td>1.108</td>
<td>2.007</td>
<td>2.094</td>
</tr>
<tr>
<td>SFP_3,4</td>
<td>1.108</td>
<td>2.074</td>
<td>2.067</td>
</tr>
<tr>
<td>SFP_3,5</td>
<td>1.108</td>
<td>2.013</td>
<td>2.019</td>
</tr>
<tr>
<td>SFP_4,5</td>
<td>0.376</td>
<td>0.555</td>
<td>0.779</td>
</tr>
<tr>
<td>SFP_1,2,3</td>
<td>1.108</td>
<td>2.154</td>
<td>2.212</td>
</tr>
<tr>
<td>SFP_1,2,4</td>
<td>1.089</td>
<td>2.154</td>
<td>2.226</td>
</tr>
<tr>
<td>SFP_1,2,5</td>
<td>1.108</td>
<td>2.007</td>
<td>2.094</td>
</tr>
<tr>
<td>SFP_1,3,4</td>
<td>1.108</td>
<td>2.074</td>
<td>2.067</td>
</tr>
<tr>
<td>SFP_1,3,5</td>
<td>1.108</td>
<td>2.013</td>
<td>2.019</td>
</tr>
<tr>
<td>SFP_1,4,5</td>
<td>0.376</td>
<td>0.555</td>
<td>0.779</td>
</tr>
<tr>
<td>SFP_2,3,4</td>
<td>0.432</td>
<td>0.605</td>
<td>0.764</td>
</tr>
<tr>
<td>SFP_2,3,5</td>
<td>0.432</td>
<td>0.605</td>
<td>0.764</td>
</tr>
<tr>
<td>SFP_2,4,5</td>
<td>0.432</td>
<td>0.605</td>
<td>0.764</td>
</tr>
<tr>
<td>SFP_3,4,5</td>
<td>0.376</td>
<td>0.555</td>
<td>0.779</td>
</tr>
<tr>
<td>SFP_1,2,3,4</td>
<td>0.031</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>SFP_1,2,3,5</td>
<td>0.031</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>SFP_1,3,4,5</td>
<td>0.031</td>
<td>0.006</td>
<td>0.006</td>
</tr>
</tbody>
</table>

**Table 4.** Results of former three-order JFP ($\times 10^{-4}$).

<table>
<thead>
<tr>
<th>JFP</th>
<th>JFP_1</th>
<th>JFP_2</th>
<th>JFP_3</th>
<th>JFP_4</th>
<th>JFP_5</th>
<th>JFP_1,2</th>
<th>JFP_1,3</th>
<th>JFP_1,4</th>
<th>JFP_1,5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.837</td>
<td>1.132</td>
<td>0.432</td>
<td>0.359</td>
<td>0.031</td>
<td>0.376</td>
<td>0.033</td>
<td>0.238</td>
<td>0.019</td>
<td></td>
</tr>
<tr>
<td>JFP_2,3</td>
<td>JFP_2</td>
<td>JFP_3</td>
<td>JFP_4</td>
<td>JFP_5</td>
<td>JFP_1,2</td>
<td>JFP_1,3</td>
<td>JFP_1,4</td>
<td>JFP_1,5</td>
<td></td>
</tr>
<tr>
<td>0.176</td>
<td>0.154</td>
<td>0.022</td>
<td>0.006</td>
<td>0.022</td>
<td>0.001</td>
<td>0.033</td>
<td>0.123</td>
<td>0.015</td>
<td></td>
</tr>
<tr>
<td>JFP_1,3,4</td>
<td>JFP_1,3,5</td>
<td>JFP_1,4,5</td>
<td>JFP_2,3,4</td>
<td>JFP_2,3,5</td>
<td>JFP_2,4,5</td>
<td>JFP_3,4,5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>0.010</td>
<td>0.001</td>
<td>0.006</td>
<td>0.017</td>
<td>0.001</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

JFP: joint failure probability.
Table 5. Bounds of the FP of the plain truss structure estimated by the boundary theories ($\times 10^{-4}$).

<table>
<thead>
<tr>
<th></th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ditlve's bounds</td>
<td>2.735</td>
<td>2.913</td>
</tr>
<tr>
<td>Zhang's bounds</td>
<td>2.930</td>
<td>3.172</td>
</tr>
<tr>
<td>Song's bounds</td>
<td>First order</td>
<td>2.700</td>
</tr>
<tr>
<td></td>
<td>Third order</td>
<td>2.761</td>
</tr>
<tr>
<td>Small-scale LP based Bounds</td>
<td>Second order</td>
<td>2.510</td>
</tr>
<tr>
<td></td>
<td>Third order</td>
<td>2.838</td>
</tr>
</tbody>
</table>

FP: failure probability and LP: linear programming.

Table 6. Distribution parameters of input variables of example 2.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean value</td>
<td>20</td>
<td>28</td>
<td>35</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>Coefficient of variation</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 7. Results of SFP’s estimated by FOSM, ISM, and MCS.

<table>
<thead>
<tr>
<th>SFPs</th>
<th>SFP1</th>
<th>SFP2</th>
<th>SFP3</th>
<th>SFP4</th>
<th>SFP5</th>
<th>SFP1,2</th>
<th>SFP1,3</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOSM</td>
<td>0.0143</td>
<td>0.0193</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0096</td>
<td>0.0193</td>
<td>0.0200</td>
</tr>
<tr>
<td>ISM</td>
<td>0.0146</td>
<td>0.0194</td>
<td>0.0200</td>
<td>0.0268</td>
<td>0.0098</td>
<td>0.0222</td>
<td>0.0229</td>
</tr>
<tr>
<td>MCS</td>
<td>0.0143</td>
<td>0.0192</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0096</td>
<td>0.0232</td>
<td>0.0239</td>
</tr>
<tr>
<td></td>
<td>SFP1,4</td>
<td>SFP1,5</td>
<td>SFP2,3</td>
<td>SFP2,4</td>
<td>SFP2,5</td>
<td>SFP3,4</td>
<td>SFP3,5</td>
</tr>
<tr>
<td>FOSM</td>
<td>0.0264</td>
<td>0.0143</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0193</td>
<td>0.0264</td>
<td>0.0200</td>
</tr>
<tr>
<td>ISM</td>
<td>0.0293</td>
<td>0.0199</td>
<td>0.0181</td>
<td>0.0339</td>
<td>0.0195</td>
<td>0.0328</td>
<td>0.0202</td>
</tr>
<tr>
<td>MCS</td>
<td>0.0293</td>
<td>0.0210</td>
<td>0.0172</td>
<td>0.0379</td>
<td>0.0198</td>
<td>0.0344</td>
<td>0.0204</td>
</tr>
<tr>
<td></td>
<td>SFP4,5</td>
<td>SFP1,2,3</td>
<td>SFP1,4,5</td>
<td>SFP2,5,4</td>
<td>SFP3,4,5</td>
<td>SFP1,4,5</td>
<td></td>
</tr>
<tr>
<td>FOSM</td>
<td>0.0264</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0193</td>
<td>0.0264</td>
<td>0.0200</td>
<td></td>
</tr>
<tr>
<td>ISM</td>
<td>0.0245</td>
<td>0.0256</td>
<td>0.0386</td>
<td>0.0248</td>
<td>0.0377</td>
<td>0.0266</td>
<td></td>
</tr>
<tr>
<td>MCS</td>
<td>0.0244</td>
<td>0.0260</td>
<td>0.0395</td>
<td>0.0259</td>
<td>0.0380</td>
<td>0.0264</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SFP2,3,4</td>
<td>SFP2,3,5</td>
<td>SFP2,4,5</td>
<td>SFP3,4,5</td>
<td>SFP1,2,4,5</td>
<td>SFP1,3,4,5</td>
<td></td>
</tr>
<tr>
<td>FOSM</td>
<td>0.0200</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0264</td>
<td>0.0200</td>
<td>0.0264</td>
<td></td>
</tr>
<tr>
<td>ISM</td>
<td>0.0341</td>
<td>0.0223</td>
<td>0.0377</td>
<td>0.0371</td>
<td>0.0351</td>
<td>0.0368</td>
<td></td>
</tr>
<tr>
<td>MCS</td>
<td>0.0368</td>
<td>0.0227</td>
<td>0.0370</td>
<td>0.0352</td>
<td>0.0355</td>
<td>0.0372</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SFP1,2,3,5</td>
<td>SFP1,2,3,5,4</td>
<td>SFP1,2,3,4,5</td>
<td>Simulation times</td>
<td>N</td>
<td>$\mu^S$</td>
<td></td>
</tr>
<tr>
<td>FOSM</td>
<td>0.0200</td>
<td>0.0200</td>
<td>0.0200</td>
<td>–</td>
<td>576</td>
<td>$6 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>ISM</td>
<td>0.0290</td>
<td>0.0383</td>
<td>0.0401</td>
<td>1000</td>
<td>31576</td>
<td>$1.5810 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>MCS</td>
<td>0.0294</td>
<td>0.0389</td>
<td>0.0393</td>
<td>$10^2$</td>
<td>5 $\times 10^7$</td>
<td>$1.0361 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

SFP: separate failure probability; FOSM: first-order second-moment method; ISM: importance sampling method; and MCS: Monte Carlo simulation.

Table 8. Results of the former three-order JFPs.

<table>
<thead>
<tr>
<th>JFP1</th>
<th>JFP2</th>
<th>JFP3</th>
<th>JFP4</th>
<th>JFP5</th>
<th>JFP1,2</th>
<th>JFP1,3</th>
<th>JFP1,4</th>
<th>JFP1,5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0143</td>
<td>0.0193</td>
<td>0.0200</td>
<td>0.0264</td>
<td>0.0096</td>
<td>0.0052</td>
<td>0.0052</td>
<td>0.0057</td>
<td>0.0014</td>
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</table>

<table>
<thead>
<tr>
<th>JFP2,3</th>
<th>JFP2,4</th>
<th>JFP2,5</th>
<th>JFP3,4</th>
<th>JFP3,5</th>
<th>JFP4,5</th>
<th>JFP1,2,3</th>
<th>JFP1,2,4</th>
<th>JFP1,2,5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0110</td>
<td>0.0039</td>
<td>0.0045</td>
<td>0.0060</td>
<td>0.0046</td>
<td>0.0058</td>
<td>0.0038</td>
<td>0.0023</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>JFP1,3,4</th>
<th>JFP1,3,5</th>
<th>JFP1,4,5</th>
<th>JFP2,3,4</th>
<th>JFP2,3,5</th>
<th>JFP2,4,5</th>
<th>JFP3,4,5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0028</td>
<td>0.0012</td>
<td>0.0013</td>
<td>0.0032</td>
<td>0.0035</td>
<td>0.0025</td>
<td>0.0030</td>
</tr>
</tbody>
</table>

JFP: joint failure probability.
where $x_i (i = 1, 2, 3, 4, 5)$ are five independently normal random variables with distribution parameters presented in Table 6.

The results of SFP by FOSM, ISM, and MCS are presented in Table 7. Due to the addition laws of FP for parallel system

$$P_p^{(5)} = \frac{1}{16} \left\{ \sum_{i=1}^{5} SFP_i - \sum_{i \neq j} SFP_{i,j} + \sum_{i \neq j \neq k} SFP_{i,j,k} - \sum_{i \neq j \neq k \neq m} SFP_{i,j,k,m} + SFP_{1,2,3,4,5} \right\}$$  (66)

Substituting the results of SFP into equation (66), the FP of the system, i.e. $P_P^{(5)}$, can be obtained which is presented in the last column of Table 7. The result of $P_P^{(5)}$ by direct MCS with $10^7$ simulations is $1.059 \times 10^{-3}$. It is seen that the result of $P_P^{(5)}$ by FOSM, i.e. $6 \times 10^{-4}$, is less than the accurate result. This error may result from the not accurate estimates for high order SFPs by FOSM. Similarly, we can yield the estimates of initial three-order JFP by substituting the results of the initial three-order SFPs into equations (32) and (34), and the results are presented in Table 8.

With the constraint of initial two- or three-order JFPs, Song’s bounds and the small-scale LP-based bounds may be used, but as mentioned in the article, the computational burden can be tremendously heavy. In response to this problem, a new LP-based boundary theory is proposed. The computational cost of this new method increases linearly with the number of failure modes, thus it is more efficient.

Theoretically, the LP model based boundary theories can be used for computing the FP of both the series and parallel system with large number of failure modes. However, as indicated in example 2, this method cannot provide a good estimate for the lower bounds of the FP of the parallel system. Considering the fact that in most cases the upper bounds of the FP of the parallel system being meaningful in engineering application, we carry out a propositional upper bound. The numerical example shows that the propositional upper bound estimated by the addition law of the FP is very close to the accurate FP of the parallel system. For general systems, Song’s bounds may be used, but as mentioned in the article, the computational burden can be tremendously heavy. Thus further study is required for general systems.

### Conclusion

In this presentation the addition laws of the FP for both series and parallel systems are proposed and proved mathematically which provides a simple way to evaluate the FP of structural system with multiple failure modes. The derived addition laws can be employed to deal with both series and parallel systems with a small number of failure modes. When dealing with structural system with a large number of failure modes, the number of the SFPs required to be estimated is too large, thus it is computationally expensive.

For the problem with large number failure modes, three widely used boundary theories are reviewed and improved by the use of the derived addition laws. Among these three boundary theories, the LP model based one has several definite advantages. The main drawback of this method is that the computation cost increases exponentially with respect to the number of failure modes. In response to this problem, a new LP-based boundary theory is proposed. The computational cost of this new method increases linearly with the number of failure modes, thus it is more efficient.

### Funding

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### Table 9. Bounds of the FP of the parallel system.

<table>
<thead>
<tr>
<th>Bound</th>
<th>Lower</th>
<th>Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Song’s bound</td>
<td>2.390 $\times 10^{-9}$</td>
<td>1.4434 $\times 10^{-3}$</td>
</tr>
<tr>
<td>Small-scale LP based bounds</td>
<td>7.2629 $\times 10^{-9}$</td>
<td>5.3477 $\times 10^{-3}$</td>
</tr>
</tbody>
</table>

FP: failure probability and LP: linear programming.
Foundation of China (2011XW010001) and Aeronautical Science Foundation of China (2011ZA53015).

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References

Appendix 1
Notation

\[ A_i \] domain where there are \( i \) modes and only \( i \) modes fail

\[ F_i \] failure domain of the \( i \)th failure mode

\[ F_{i}^{(n)} \] failure domain of a parallel structural system with \( n \) failure modes

\[ F_{S}^{(n)} \] failure domain of a series structural system with \( n \) failure modes

\[ JFP_{i_{1}, i_{2}, \ldots, i_{p}} \] \( p \)th order JFP

\[ P_i \] probability measure of \( A_i \)

\[ PF_i \] FP of the \( i \)th failure mode

\[ P_{i}^{(n)} \] FP of a parallel structural system with \( n \) failure modes

\[ P_{S}^{(n)} \] FP of a series structural system with \( n \) failure modes

\[ SFP_{i_{1}, i_{2}, \ldots, i_{p}} \] \( p \)th order SFP

\[ TFP_{i_{1}, i_{2}, \ldots, i_{p}} \] \( p \)th order TFP

Appendix 2
Proof of the addition laws of FP

Before the proof, two formulas should be introduced first.

Formula 1: The addition law of probability

\[
P_{S}^{(n+1)} = \sum_{i=1}^{n+1} JFP_{i} - \sum_{i \neq j} JFP_{i,j} + \sum_{i \neq j \neq k} JFP_{i,j,k} - \cdots (-1)^{n} JFP_{1,2,\ldots,n+1}
\]  
(67)

Formula 2:

\[
P_{S}^{(n+1)} = \frac{1}{n+1} \left[ \sum_{i=1}^{n+1} \left( \sum_{k=1, k \neq i}^{n+1} F_{k} \right) + P_{1} \right]
\]  
(68)

where \( P_{1} \) denotes the probability there is one mode and only one mode fails. Equation (67) is the well-known addition law of probability in probability theory. Equation (68) is a new formula, thus the proof need to be carried out.

Since

\[
\bigcup_{i=1}^{n+1} F_{i} = \bigcup_{i=2}^{n+1} \left( \bigcup_{i=2}^{n+1} F_{i} \cap F_{i} \right)
\]

\[= \bigcup_{i=1}^{n+1} F_{i} \quad \text{and} \quad \left( \bigcup_{i=2}^{n+1} F_{i} \right) \cap \left( \bigcup_{i=2}^{n+1} F_{i} \cap F_{i} \right) = \emptyset,
\]
then
\[ P_S^{(n+1)} = P \left( \bigcup_{i=1}^{n+1} F_i \right) = P \left( \bigcup_{i=1}^{n+1} F_i \right) + P \left( \bigcup_{i=2}^{n+1} F_i \cap F_1 \right) \] (69)

Similarly,
\[ P_S^{(n+1)} = P \left( \bigcup_{i=1}^{n+1} F_i \right) = P \left( \bigcup_{i=1}^{n+1} F_i \right) + P \left( \bigcup_{i=1}^{n+1} F_i \cap F_{n+1} \right) \] (70)

Combining equations (69) and (70) yields
\[ (n + 1)P_S^{(n+1)} = \sum_{i=1}^{n+1} P \left( \bigcup_{k=1, k \neq i}^{n+1} F_k \right) + \sum_{i=1}^{n+1} P \left( \bigcup_{k=1, k \neq i}^{n+1} F_k \cap F_i \right) \] (71)

Apparentlty, the second-order of the right side of the equation (71) equals \( P_1 \), i.e.
\[ P_1 = \sum_{i=1}^{n+1} P \left( \bigcup_{k=1, k \neq i}^{n+1} F_k \cap F_i \right) \] (72)

Thus equation (68) holds.

After preparing this work, we prove the addition laws of FP by mathematical induction method. First, we should prove they are true for the case of two failure modes.

From equation (67), we can obtain the following equality
\[ P_S^{(2)} = JFP_1 + JFP_2 - JFP_{1,2} \] (73)

And by use of equation (68) in the case \( n + 1 = 2 \), we obtain equation (74)
\[ P_S^{(2)} = \frac{1}{2} \{ JFP_1 + JFP_2 + P_1 \} \] (74)

Obviously, for the case with two failure modes, \( P_1 \) equals the SFP of these two failure modes, i.e.
\[ P_1 = SFP_{1,2} \] (75)

Substituting equation (75) into equation (74) yields
\[ P_S^{(2)} = \frac{1}{2} \{ JFP_1 + JFP_2 + SFP_{1,2} \} \] (76)

Subtracting equation (73) from equations (76) and (77) holds
\[ JFP_1 + JFP_2 - JFP_{1,2} - \frac{1}{2} \{ JFP_1 + JFP_2 + SFP_{1,2} \} = 0 \] (77)

By simplifying equation (77) we have equation (78)
\[ JFP_{1,2} = \frac{1}{2} \{ JFP_1 + JFP_2 - SFP_{1,2} \} \] (78)

Substituting equation (78) into equation (73) yields
\[ P_S^{(2)} = \frac{1}{2} \{ JFP_1 + JFP_2 + SFP_{1,2} \} \] (79)

Considering the fact that \( JFP_i = SFP_i \) and \( JFP_{1,2} = P_1 \), we prove that the addition laws of FP in the case of two failure modes are correct
\[ P_S^{(2)} = \frac{1}{2} \{ JFP_1 + JFP_2 + SFP_{1,2} \} \] (80)
\[ P_P^{(2)} = \frac{1}{2} \{ JFP_1 + JFP_2 - SFP_{1,2} \} \] (81)

Equations (80) and (81) are the addition laws of FP for series and parallel systems with two failure modes, respectively. Next we assume that the addition laws are true for system with \( n \) failure modes, the next work is to prove that the addition laws hold true for systems with \( n + 1 \) failure modes.

In equation (67) with \( n + 1 \) failure modes, all the former \( n \) order JFPs can be expressed with the former \( n \) order SFPs, thus equation (82) holds
\[ P_S^{(n+1)} = \sum_{i=1}^{n+1} SFP_i - \frac{1}{2} \sum_{i \neq j}^{n+1} [SFP_i + SFP_j - SFP_{ij}] \]
\[ + \cdots + (-1)^{n-1} \frac{1}{2^{n-2}} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} SFP_{ij} \]
\[ - \sum_{j \neq k} SFP_{ij,k} + \sum_{j \neq k} SFP_{ij,k,ik} \]
\[ + \cdots + (-1)^{n-1} SFP_{ij,k,ik,...} + (-1)^n JFP_{1,2,...,n+1} \] (82)
Incorporating the similar terms in equation (82) yields

\begin{align*}
p^{(n+1)}_S &= \left[ \sum_{i=1}^{n} (-1)^{i+1-2} \binom{n}{i} C_1^{C_n+1} \right] \sum_{i=1}^{n+1} SFP_i \\
&+ \left[ \sum_{i=2}^{n} (-1)^{i+2-2} \binom{n}{i} C_1^{C_n+1} \right] \sum_{i \neq j} SFP_{i,j} + \cdots \\
&+ \left[ \sum_{i=p}^{n} (-1)^{i+p-2} \binom{n}{i} C_1^{C_n+1} \right] \times \sum_{i_1 \neq i_2 \neq \cdots \neq i_p} SFP_{i_1,i_2,\ldots,i_p} + \cdots \\
&+ \left( -1 \right)^{n+p-2} \binom{n}{n} C^{C_n+1} \times \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} SFP_{i_1,i_2,\ldots,i_n} \\
&+ \left(-1\right)^{p} JFP_{1,2,\ldots,n+1}
\end{align*}

(83)

The factor of each item is a summation of series which can be solved by the technique of series summation. In this article these summations of series are given directly as follows

\begin{equation}
\sum_{i=p}^{n} (-1)^{i+p-2} \binom{n}{i} C^{C_n+1} = \frac{1 + (-1)^{n+p}}{2^n}
\end{equation}

(84)

Substituting equation (84) into equation (83) yields

\begin{align*}
p^{(n+1)}_S &= \sum_{i=1}^{n} \left[ 1 + (-1)^{i+p} \frac{SFP_{i_1,i_2,\ldots,i_p}}{2^n} \right] \\
&+ \left(-1\right)^{p} JFP_{1,2,\ldots,n+1}
\end{align*}

(85)

If \( n + 1 \) is an odd number

\begin{align*}
p^{(n+1)}_S &= \frac{1}{2^{n-1}} \left[ \sum_{i_1 \neq i_2} SFP_{i_1,i_2} + \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} SFP_{i_1,i_2,i_3,i_4} \\
&+ \cdots + \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} SFP_{i_1,i_2,\ldots,i_n} \right] + JFP_{1,2,\ldots,n+1}
\end{align*}

(86)

If \( n + 1 \) is an even number

\begin{align*}
p^{(n+1)}_S &= \frac{1}{2^{n-1}} \left[ \sum_{i=1}^{n+1} SFP_i + \sum_{i_1 \neq i_2 \neq i_3} SFP_{i_1,i_2,i_3} + \cdots \\
&+ \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} SFP_{i_1,i_2,i_3,i_4} \right] - JFP_{1,2,\ldots,n+1}
\end{align*}

(87)

Here we need to introduce a set of equations, which are shown as follows

\begin{align*}
P_{n+1} &= JFP_{1,2,\ldots,n+1} \\
C_{n+1}^n P_{n+1} + P_n &= \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} JFP_{i_1,i_2,\ldots,i_n} \\
C_{n+1}^{n-1} P_{n+1} + \frac{C_{n-1}^1}{C_{n+1}^n} C_{n-1}^n P_n + P_{n-1} &= \sum_{i_1 \neq i_2 \neq \cdots \neq i_{n-1}} JFP_{i_1,i_2,\ldots,i_{n-1}} \\
C_{n+1}^{n-2} P_{n+1} + \frac{C_{n-2}^2}{C_{n+1}^n} C_{n-2}^n P_n + \frac{C_{n-2}^1}{C_{n+1}^{n-1}} C_{n-2}^{n-1} P_{n-1} + P_{n-2} &= \sum_{i_1 \neq i_2 \neq \cdots \neq i_{n-2}} JFP_{i_1,i_2,\ldots,i_{n-2}} \\
\vdots \\
C_{n+1}^n P_{n+1} + \frac{C_{n-1}^1}{C_{n+1}^n} C_{n-1}^n P_n + \frac{C_{n-1}^1}{C_{n+1}^{n-1}} C_{n-1}^{n-1} P_{n-1} + \frac{C_{n-1}^1}{C_{n+1}^{n-2}} P_{n-2} \cdots + P_1 &= \sum_{i=1}^{n+1} JFP_i
\end{align*}

(88)

where \( P_i (i = 0, 1, \ldots, n + 1) \) denotes the probability there are \( i \) modes and only \( i \) modes fail. This equation set can be explained as follows: \( JFP_{1,2,\ldots,n+1} \) equals the FP of the system with all of the \( n + 1 \) modes fail which is just what the first equation indicates. \( \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} JFP_{i_1,i_2,\ldots,i_n} \) is a summation of \( C_{n+1}^n \) orders, each of which equals the probability with \( n \) modes fails. If the last one does not fail then this summation equals \( P_n \). If the last one fails, then every order of this summation equals \( P_{n+1} \), thus this summation equals \( C_{n+1}^n P_{n+1} \). This is how the second equation is obtained. The other \( n - 1 \) equations can be obtained in the similar manner.

If \( n + 1 \) is an odd number, by solving this equations set (the solving process is left out), we have

\begin{align*}
P_3 + P_5 + \cdots P_{n+1} &= \left( 2^n - n - 1 \right) JFP_{1,2,\ldots,n+1} \\
&- \frac{2^n}{2} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} JFP_{i_1,i_2,\ldots,i_n} \\
&+ \frac{2^n}{2} \sum_{i_1 \neq i_2 \neq \cdots \neq i_{n-1}} JFP_{i_1,i_2,\ldots,i_{n-1}} \cdots + \sum_{i_1 \neq i_2 \neq i_3} JFP_{i_1,i_2,i_3}
\end{align*}

(89)
For the case that \( n + 1 \) is an odd number

\[
SFP_{1,2,\ldots,n+1} = P_1 + P_3 + \ldots + P_{n-1} + P_{n+1}
\]

From equation (68)

\[
P^{(n+1)}_S = \frac{1}{n+1} \left[ \sum_{i=1}^{n+1} P \left( \bigcup_{k=1, k \neq i}^{n+1} F_k \right) + P_1 \right]
\]

= \frac{1}{n+1} \left[ \sum_{i=1}^{n+1} P \left( \bigcup_{k=1, k \neq i}^{n+1} F_k \right) + SFP_{1,2,\ldots,n+1} \right.

- \left( P_3 + P_5 + \ldots + P_{n+1} \right) \right] \quad (91)

Substitute equation (89) into equation (91) yields

\[
P^{(n+1)}_S = \frac{1}{n+1} \left[ -(2^n - n - 1)JFP_{1,2,\ldots,n+1} + \right.

+ JFP_{1,2,\ldots,n+1} + \left( -1 + \frac{2^n}{2} - n \right)

\times \sum_{i_1 \neq i_2 \neq \ldots \neq i_{n+1}} JFP_{i_1,i_2,\ldots,i_n} \right.

+ \left( \frac{C_{n+1}^n C_{n+1}^{n-2}}{C^{n-1}_{n+1}} - \frac{2^n}{2} + n - 1 \right)

\times \sum_{i_1 \neq i_2 \neq \ldots \neq i_{n+1}} JFP_{i_1,i_2,\ldots,i_{n+1}} + \ldots

\left. + \left( \frac{C_{n+1}^n C_n^3}{C_{n+1}^3} - 1 \right) \sum_{i_1 \neq i_2 \neq i_3} JFP_{i_1,i_2,i_3} \right]

\left. - \frac{C_{n+1}^n C_n^2}{C_{n+1}^2} \sum_{i_1 \neq i_2} JFP_{i_1,i_2} + \frac{C_{n+1}^n C_n^2}{C_{n+1}^2} \sum_{i_1 \neq i_2} JFP_{i_1,i_2} \right] \quad (92)

Take the place of the former \( n \) order JFP in equation (92) with their expressions in the form of SFP and simplify the obtained equation, we have

\[
P^{(n+1)}_S = \frac{1}{n+1} \left[ -(2^n - n - 1)JFP_{1,2,\ldots,n+1} \right.

+ SFP_{1,2,\ldots,n+1} \left( 1 - \frac{n+1}{2^n - 1} \right)

\times \sum_{i_1 \neq i_2 \neq \ldots \neq i_n} SFP_{i_1,i_2,\ldots,i_n} \right]
\]

Thus

\[
P^{(n+1)}_S = \frac{1}{2^n} \left[ \sum_{i=1}^{n+1} SFP_i + \sum_{i_1 \neq i_2} SFP_{i_1,i_2} + \ldots + SFP_{i_1,i_2,\ldots,i_{n+1}} \right] \quad (95)
\]

By substituting equation (95) into equation (86) and simplifying the obtained equation, we have

\[
P^{(n+1)}_F = \frac{1}{2^n} \left[ \sum_{i=1}^{n+1} SFP_i - \sum_{i_1 \neq i_2} SFP_{i_1,i_2} + \ldots \right.

\left. + (-1)^n SFP_{i_1,i_2,\ldots,i_{n+1}} \right] \quad (96)
\]

Equations (95) and (96) are exactly the addition laws of FP of series and parallel system with \( n + 1 \) failure modes, respectively for the case that \( n + 1 \) is an odd number. For the case that \( n + 1 \) is an even number, the similar process can be performed and the similar result can be obtained. Till now we have proved that if the addition laws are true for the system with \( n \) failure modes, then they must be true for the system with \( n + 1 \) failure modes. Thus, the proof of the addition laws of FP is complicated.