Adaptive fuzzy quasi-continuous high-order sliding mode controller for output feedback tracking control of robot manipulators

Van Mien¹, Hee-Jun Kang² and Kyoo-Sik Shin³

Abstract
This article develops a new output feedback tracking control scheme for uncertain robot manipulators with only position measurements. Unlike the conventional sliding mode controller, a quasi-continuous second-order sliding mode controller (QC2C) is first designed. Although the QC2C produces continuous control and less chattering than conventional sliding mode and other high-order sliding mode controllers, chattering exists when the sliding manifold is defined by the equation $s = \dot{s} = 0$. To alleviate the chattering, an adaptive fuzzy QC2C (FQC2C) is designed, in which the fuzzy system is used to adaptively tune the sliding mode controller gain. Furthermore, in order to eliminate chattering and achieve higher tracking accuracy, quasi-continuous third-order sliding mode controller (QC3C) and fuzzy QC3C (FQC3C) are investigated. These controllers incorporate a super-twisting second-order sliding mode observer for estimating the joint velocities, and a robust exact differentiator to estimate the sliding manifold derivative; therefore, the velocity measurement is not required. Finally, computer simulation results for a PUMA560 industrial robot are also shown to verify the effectiveness of the proposed strategy.

Keywords
Robot manipulators, output feedback tracking control, quasi-continuous high-order sliding mode, fuzzy sliding mode controller

Introduction
Various controller schemes have been developed for robotic manipulators, including PID control,¹ computed torque control,²,³ adaptive control⁴ and fuzzy control.⁵ Most of them are based on the assumption that a complete state measurement (position and velocity) of the robot manipulator is available. The position measurement can be accurately obtained by the encoders. In contrast, velocity measurements are obtained using tachometers which are often contaminated by noise thereby reducing the control performance of these methods. Therefore, it is important to investigate robot controllers with only joint position measurements. As an outcome of the research directed at this topic, several output feedback tracking control schemes have been developed based on several velocity observer schemes.⁶–⁸ However, these approaches cannot provide exact finite time convergence of the state derivatives and motion tracking. To obtain the finite time convergence of the state derivatives, a supper-twisting second-order sliding mode (SOSM) observer has been developed.⁹,¹⁰

As sliding mode control (SMC) is robust with respect to system uncertainties and has a fast transient response, it has received a great deal of attention from the research community.¹¹–¹³ The main idea of SMC is to design a sliding surface first and then to design a control law that forces the system’s state to reach and remain on the sliding manifold, which is designed to achieve control objectives. However, the major drawback of SMC in practical applications is undesired chattering due to high-frequency switching. Numerous techniques have been proposed to avoid the dangerous chattering of SMC. The sign function

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can be replaced in a small area of the surface by a smooth approximation, which is the so-called boundary layer control that implies deterioration accuracy and robustness.\textsuperscript{12,13} Another approach is to use a high-order sliding mode (HOSM) controller that yields less chattering and better convergence accuracy with respect to the conventional SMC.\textsuperscript{9,10,14–20} Unlike the classical SMC, which works on the first time derivative of the sliding variable, HOSM works with the discontinuous control acting on the higher order time derivative. By moving the switching to the higher derivatives of the control, chattering reduction is achieved because the control signal is now continuous. Among HOSM controllers, quasi-continuous HOSM controller-based robust exact differentiators have been developed and applied successfully for real applications.\textsuperscript{15,18–20} The major advantage of the quasi-continuous HOSM is that it produces continuous control and produces less chattering than other HOSM controllers.\textsuperscript{20} However, when the system exists in the sliding manifold $\dot{x} = \dot{s} = \cdots = \dot{s}^{(r-1)} = 0$ of the $r$-sliding mode, significant chattering is generated due to the presence of switching delays, measurement noises and singular perturbations.

In recent years, fuzzy logic has been widely applied as a successful practical approach in control fields to control or model control systems with uncertainties.\textsuperscript{21–23} One major feature of fuzzy logic is its capability to express human thinking. Thus, fuzzy logic controls, in general, are suitable for many industrial systems that cannot be precisely described by mathematical formulations. Because of the advantages of SMC and fuzzy control, many approaches have been developed to combine them.\textsuperscript{24–29} These approaches can be classified into three major approaches to incorporate fuzzy logic control into SMC.

1. Fuzzy logic is used to represent the unknown dynamics and the SMC is then applied to stabilize the fuzzy system.\textsuperscript{24,25}
2. Fuzzy logic control is used to replace the discontinuous $\text{sign}$ function of the reaching law in conventional SMC to avoid chattering.\textsuperscript{26–28}
3. Fuzzy logic control is applied to adaptively tune the sliding mode gain of conventional SMC.\textsuperscript{26,29–31} In this case, the robustness property of the sliding mode is preserved but the chattering is alleviated.

In light of the remarkable benefits, in this article, an output feedback tracking control scheme for uncertain robot manipulators is developed based on a quasi-continuous HOSM control-based exact differentiator and a super-twisting SOSM observer. To overcome the drawback of conventional SMC, a quasi-continuous SOSM which provides a continuous signal and less chattering is first designed. Then, a fuzzy logic system is employed to adaptively tune the sliding mode gain in order to minimize chattering during the control action of quasi-continuous second-order sliding mode controller (QC2C) when the system remains on the sliding manifold. In addition, to further eliminate chattering and obtain higher accuracy, a quasi-continuous third-order sliding mode controller (QC3C) is investigated. However, the major drawback of the QC3C compared to the QC2C is that the tracking response is too slow. To overcome this obstacle, a fuzzy QC3C (FQC3C) is designed.

The remainder of this article is organized as follows: in the following section, the proposed formulation is presented. In the next section, a fuzzy QC2C (FQC2C) is designed. The design of the FQC3C is described in the subsequent section followed by computer simulation results on a PUMA560 robot in the following section. The last section outlines some conclusions.

**Problem statements**

According to Lagrange theory,\textsuperscript{2} the dynamic equation of an $n$-link robot manipulator can be described by

$$\tau = M(q)\ddot{q} + V_m(q, \dot{q})\dot{q} + G(q) + f(q, \dot{q})$$  \hspace{1cm} (1)

where $q \in \mathbb{R}^n$ is the state vector, $\tau \in \mathbb{R}^n$ is the torque produced by actuators, $M(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $V_m(q, \dot{q}) \in \mathbb{R}^n$ is the Coriolis and centripetal torque, $f(q, \dot{q}) \in \mathbb{R}^n$ is unmodeled dynamics including friction terms and external disturbances and so on and $G(q) \in \mathbb{R}^n$ is the gravity torque term.

Equation (1) can be rewritten as

$$\ddot{q} = M^{-1}(q)[\tau - V_m(q, \dot{q})\dot{q} - G(q) - f(q, \dot{q})]$$   \hspace{1cm} (2)

Using $x_1 = q \in \mathbb{R}^n$ and $x_2 = \dot{q} \in \mathbb{R}^n$, the robot dynamics can be described in state space form as

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= h(x_1, x_2, \tau) + \Delta(x_1, x_2, t) \\
y &= x_1
\end{align*}$$   \hspace{1cm} (3)

where $h(x_1, x_2, \tau) = M^{-1}(q)[\tau - V_m(q, \dot{q})\dot{q} - G(q)]$ represents the nominal system, and $\Delta(q, \dot{q}, t) = M^{-1}(q)[-f(q, \dot{q})]$ represents the modeling uncertainty in the dynamic model of robot manipulators.

The objective of this article is to design a controller scheme that gets the real trajectory $q(t)$ to track the specific desired trajectory $q_d(t)$. In other words, it is required to keep the motion tracking error asymptotically at zero in spite of the parameter uncertainties, external perturbations and measurement errors. To do this, conventional sliding mode controllers have been designed. The design procedure consists of two main steps. The first step involves constructing the desired sliding surface, which is chosen such that when it converges to zero, the desired control is
achieved. The next step is to select a control law that forces the system state to reach the sliding surface in a finite time. For the first step, we let

\[ e = x_1 - x_d \]

(4)

where \( x_d = q_d \) is the desired trajectory. Then, the sliding surface is selected as

\[ s = \dot{e} + \beta e \]

(5)

where \( \beta \) is a strictly positive constant. It is obvious that, if the system state remains on the sliding surface, the desired control task \( e \to 0 \) can be achieved.

From equation (3), the robot dynamics can be rewritten as

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = M^{-1}(x_1)u + g(x_1, x_2) + \Delta(x_1, x_2, t) \]

(6)

where \( u = \tau \) is the control input, \( g(x_1, x_2) = M^{-1}(x_1)[-V_m(x_1, x_2)x_2 - G(x_1)] \).

For the second step, to ensure that the trajectories of the system approach the sliding surface, then, in the presence of the uncertainties, the derivative of the sliding surface \( \dot{s} = 0 \) should be satisfied such that

\[ \dot{s} = \dot{\varepsilon} + \beta \varepsilon \]
\[ = M^{-1}(x_1)u + g(x_1, x_2) + \Delta(x_1, x_2, t) \]
\[ - \dot{x}_d + \beta(x_2 - \dot{x}_d) \]

(7)

According to the sliding mode design procedure, we choose

\[ u = u_{eq} + u_c \]

(8)

where

\[ u_{eq} = M(x_1)[\dot{x}_d - \beta(x_2 - \dot{x}_d) - g(x_1, x_2)] \]

(9)

and \( u_c \) is the term that compensates for the effect of uncertainties. It is designed such that

\[ u_c = -M(x_1)k \text{sign}(s) \]

(10)

where \( k \) is a constant chosen based on the upper bound of the uncertainties: \( k \geq \Delta \), where \( \Delta \) is the upper bound of the uncertainties \( \Delta(x_1, x_2, t) \leq \Delta \). The stability of the system under the control law (8) is given in Theorem 1.

**Theorem 1.** Consider uncertain robot manipulators with the dynamic model described by equation (6) in the presence of modeling uncertainty. The proposed controller law given by equation (8) ensures that the sliding surface \( s \) asymptotically converges to zero in a finite time.

**Proof.** Let the Lyapunov function be \( V = \frac{1}{2}s^Ts \). Differentiating \( V \) with respect to time gives

\[ \dot{V} = s^T\dot{s} = s^T[-k\text{sign}(s) + \Delta(x_1, x_2, t)] \leq s^T[-k\text{sign}(s) + \Delta] \]

(11)

If \( k \geq |\Delta| \) is satisfied, then \( \dot{V} < 0 \) holds. This means that the sliding surface \( s \) asymptotically converges to zero.

The control law in (8) has robustness and a fast response to uncertainties. However, its application has two drawbacks for real systems: (1) it requires significant control efforts and chattering which decrease the tracking performance and (2) it assumes that the velocity is measured. However, in real application, the velocity measurement is usually not available. Thus, the problem considered in this work is to propose a new output feedback tracking control scheme based on a higher order SMC to achieve higher accuracy and less chattering using only position measurement.

**Design of FQC2C**

Conventional SMC can guarantee finite time convergence of the system. However, the conventional SMC is often impractical for robot control application due to chattering and low accuracy. To overcome the drawbacks of the conventional SMC, quasi-continuous HOSM controllers have been developed,\(^{18-20}\) keeping the main feature of the conventional SMC but providing less chattering and higher accuracy. However, it is difficult to select a suitable sliding gain that minimizes the reaching time and the chattering of the control action. Thus a fuzzy quasi-HOSM controller is proposed, in which the fuzzy system is used to adaptively tune the sliding mode gain. The main purpose is to increase the tracking response when the system is far from the sliding manifold and avoid chattering when the system stays in the sliding manifold (\( s = 0 \)).

**Quasi-continuous second-order sliding mode controller**

The QC2C is designed as

\[ u = u_{eq} + u_{QC2} \]

(12)

where \( u_{eq} \) is designed similar to equation (9). Because the velocity measurement \( x_3 \) is not available, a super-twisting SOSM observer\(^6\) is applied to estimate the velocities

\[ \dot{x}_1 = \dot{x}_2 + \alpha_1|x_1 - \hat{x}_1|^{1/2}\text{sign}(x_1 - \hat{x}_1) \]
\[ \dot{x}_2 = b(x_1, \dot{x}_2, u) + \alpha_2\text{sign}(x_1 - \hat{x}_1) \]

(13)
When there is no measurement noise and all the coefficients are properly selected as shown in Ref. 10, the state estimations converge toward the real states:

\[ \hat{x}_1 = x_1; \quad \hat{x}_2 = x_2 \]  
(14)

In this case, the equivalent control is now designed as

\[ u_{eq} = M(x)\left[\hat{x}_d - \beta(\hat{x}_2 - \hat{x}_d) - g(x_1, \hat{x}_2)\right] \]  
(15)

Based on Bartolini et al. 16 the uncertainty compensator term \( u_{QC2} \) is designed as

\[ u_{QC2} = -M(x)k\left(\frac{\dot{s} + |s|^{1/2}\text{sign}(s)}{|s| + |s|^{1/2}}\right) \]  
(16)

where \( k \) is a positive number, and it is chosen as \( k \geq \Delta \).

To design equation (16), we need to know the derivative of the sliding surface \( \dot{s} \). To obtain this, the first-order exact differentiator technique is applied: 14

\[
\begin{align*}
\dot{z}_0 &= v_0, \\
v_0 &= -\lambda_1|z_0 - s|^{1/2}\text{sign}(z_0 - s) + z_1 \\
\dot{z}_1 &= v_1 \\
v_1 &= -\lambda_2\text{sign}(z_0 - s)
\end{align*}
\]  
(17)

When all the coefficients are properly selected, the first-order sliding mode differentiator equation (17) can achieve

\[ z_0 = s; \quad z_1 = \dot{s} \]  
(18)

Substituting equation (18) into equation (16), we obtain

\[ u_{QC2} = -M(x)k\left(\frac{z_1 + |z_0|^{1/2}\text{sign}(z_0)}{|z_1| + |z_0|^{1/2}}\right) \]  
(19)

The derivative of the sliding surface under the effect of the controller law (12) is now obtained as

\[
\begin{align*}
\dot{s} &= \ddot{e} + \beta \dot{e} \\
&= M^{-1}(x_1)(u_{eq} + u_{QC2}) + g(x_1, x_2) \\
&\quad + \Delta(x_1, x_2, t) - \ddot{x}_d + \beta(x_2 - \dot{x}_d) \\
&= M^{-1}(x_1)u_{QC2} + \Delta(x_1, x_2, t)
\end{align*}
\]  
(20)

The stability of the robot system under the controller scheme in equation (12) is demonstrated in Theorem 2.

**Theorem 2.** Consider uncertain robot manipulators with the dynamic model described by equation (6). The first-order exact differentiator equation (17) is assumed to converge in finite time. The proposed controller law given by equation (19) ensures that the sliding surface \( s \) asymptotically converges to zero in a finite time.

**Proof.** Let the Lyapunov function be \( V = \frac{1}{2}s^Ts \). Differentiating \( V \) with respect to time gives

\[
\begin{align*}
\dot{V} &= s^T\dot{s} = s^T[M^{-1}(x_1)u_{QC2} + \Delta] \\
&= s^T[-k\frac{\dot{s} + |s|^{1/2}\text{sign}(s)}{|s| + |s|^{1/2}} + \Delta] \\
&= s^T\Delta - s^T\beta\text{sign}(s)\left(\frac{|s|^{1/2}}{|s| + |s|^{1/2}}\right) \\
&= |s|k\left(\Delta\text{sign}(s) \right) - |s|\left(\frac{\beta\text{sign}(s) + |s|^{1/2}}{|s| + |s|^{1/2}}\right)
\end{align*}
\]  
(21)

From equation (21), the negative of the derivative Lyapunov function is obtained when

\[ \frac{\text{sign}(s) + |s|^{1/2}}{|s| + |s|^{1/2}} \geq \frac{\Delta\text{sign}(s)}{k} \]  
(22)

is satisfied.

In the other case, we have \( \frac{\text{sign}(s) + |s|^{1/2}}{|s| + |s|^{1/2}} \leq 1 \), the condition in equation (22) can be reconstructed as

\[ k \geq \Delta\text{sign}(s) \]  
(23)

By choosing \( k \geq \Delta(\Delta \leq \Delta) \), the condition in equation (23) is satisfied, then \( \dot{V} < 0 \) holds. This means that the sliding surface \( s \) asymptotically converges to zero.

**Fuzzy quasi-continuous second-order sliding mode controller**

During the design of the QC2C in previous section, the sliding gain \( k \) in the control law of equations (12) and (19) should be chosen to be larger than the upper bound of the uncertainty in order to satisfy the existence condition of the sliding mode. A large sliding gain generates significant chattering when the sliding manifold converges to zero, \( s = \dot{s} = 0 \) consequently decreasing the system performance. In the other case, the uncertainties are often unknown and time-varying so that the sliding gain should be chosen as time-varying to reduce the chattering. For this purpose, in this section we design a modified compensated control \( u_{QC2} \). Its purpose is to adaptively tune the sliding gain \( k \) to follow the time-varying uncertainties while we guarantee the stability of the control system.

The compensated control \( u_{QC2} \) is now designed as

\[ u_{QC2} = -M(x)k\left(\frac{z_1 + |z_0|^{1/2}\text{sign}(z_0)}{|z_1| + |z_0|^{1/2}}\right) \]  
(24)
IF-THEN rules to adjust the control gain \( \hat{k} \) to be suitable with the current operating conditions (initial condition, magnitude of the uncertainties, etc.) of the controlled system. In the presence of the uncertainties, the operating conditions of the system can be illustrated as a set of points \( A_i \) in the sliding surface and the derivative of the sliding surface \( (s, \dot{s}) \) state space in Figure 2. Each \( A_1, \ldots, A_n \) represents an operating point corresponding to the magnitude of the uncertainties of the system at each time. Based on the knowledge of the controlled system, we decide the operation of the sliding gain \( \hat{k} \) corresponding to the output of the fuzzy system as follows: when the operating point \( A \) is far from the sliding surface \( (s = \dot{s} = 0) \) such as at point \( A_1 \), the control signal of QC2C is continuous. The sliding gain should be correspondingly large to force the state trajectories to reach the sliding surface rapidly. In contrast, when the system remains around the sliding surface such as at point \( A_n \), the control signal is now discontinuous. The sliding gain should be very small to reduce chattering. Therefore, the structure of the adaptive fuzzy control is implemented as follows:

The adaptive fuzzy control illustrated in Figure 1 has two inputs \( s(t), \dot{s}(t) \) and one output \( u_f(t) \). They can be defined as the following fuzzy subsets:

\[
\begin{align*}

s = \dot{s} = \{NB, NS, Z, PS, PB\}
\end{align*}
\]

and

\[
\begin{align*}

u_f = \{NVH, NH, NVB, NB, NM, NS, Z, PS, PM, PB, PVB, PH, PVH\}
\end{align*}
\]

where NVH: negative very huge; NH: negative huge; NVB: negative very big; NB: negative big; NM: negative medium; NS: negative small; Z: zero; PS: positive small; PM: positive medium; PB: positive big; PVB: positive very big; PH: positive huge; PVB: positive very huge.

Table 1. Fuzzy rule bases.

<table>
<thead>
<tr>
<th>( u_f(t) )</th>
<th>( s )</th>
<th>NB</th>
<th>NS</th>
<th>Z</th>
<th>PS</th>
<th>PB</th>
</tr>
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<tr>
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<td>NH</td>
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<td>NB</td>
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<tr>
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<td>NVB</td>
<td>NB</td>
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<td>PB</td>
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<td></td>
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<td>PVH</td>
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</tr>
</tbody>
</table>

Figure 1. Structure of the fuzzy QC2C. QC2C: Quasi-continuous second-order sliding mode controller; SOSM: second-order sliding mode.

Figure 2. Operating point of the robotic systems.

Table 1. Fuzzy rule bases.

\[
\begin{align*}

u_f(t) = \frac{\sum_{j=1}^{m} \mu_j(s(t)) \wedge \mu_j(\dot{s}(t))}{\sum_{j=1}^{m} \mu_j} \quad (27)
\end{align*}
\]

where \( h_j = \mu_j(s(t)) \wedge \mu_j(\dot{s}(t)) \) are the membership functions of the fuzzy subsets \( s_j \) and \( \dot{s}_j \) and the membership functions of the fuzzy output \( u_f(t) \) are shown in Figure 3. \( j = 1, \ldots, m \) denotes the number of fuzzy IF-THEN rules. The operator “\( \wedge \)” denotes the minimum. Then, the adaptive tune sliding gain is defined as

\[
\hat{k} = u_f(t) \quad (28)
\]
From the equations (12), (24) and (28), the FQC2C is obtained as
\[ u = u_{eq} + u_{FQC2} \]
where
\[ u_{FQC2} = M(x_1)u_{f}(\frac{z_1 + |z_0|^{1/2}\text{sign}(z_0)}{|z_1| + |z_0|^{1/2}}) \]

**Design of FQC3C**

Due to the features of the HOSM property, higher orders may provide higher accuracy and less chattering. Consequently, an FQC3C is expected to further eliminate chattering.

The quasi-continuous third-order sliding mode controller is designed as
\[ u = u_{eq} + u_{QC3} \]
where \( u_{eq} \) is designed as equation (15), and the \( u_{QC3} \) is designed as
\[ u_{QC3} = M(x_1)k\left(\frac{\dot{s} + 2(\dot{|s|} + |s|^{2/3})^{-1/2}\text{sign}(s)}{|\dot{s}| + 2(\dot{|s|} + |s|^{2/3})^{1/2}} \right) \]
where \( k \) is a positive parameter, and is chosen \( k \geq \Delta \).

The differentiation, \( s, \dot{s}, \ddot{s} \) which is used in equation (32) can be obtained by using the second-order exact differentiation:
\[ \dot{z}_0 = v_0, \]
\[ v_0 = -\lambda_1|z_0 - s|^{2/3}\text{sign}(z_0 - s) + z_1 \]
\[ \dot{z}_1 = v_1 \]
\[ v_1 = -\lambda_2|z_1 - v_0|^{1/2}\text{sign}(z_1 - v_0) + z_2 \]
\[ \dot{z}_2 = -\lambda_3\text{sign}(z_2 - v_1) \]

Using suitably chosen parameters \( \lambda_i \), the second-order exact differentiator can achieve
\[ z_0 = s, \dot{z}_1 = \dot{s}, \dot{z}_2 = \ddot{s} \]

Substituting equation (34) into equation (32) yields
\[ u_{QC3} = M(x_1)k\left(\frac{z_2 + 2(|z_1| + |z_0|)^{2/3})^{-1/2}}{|z_2| + 2(|z_1| + |z_0|)^{2/3})^{1/2}} \right) \]

The stability of the quasi-continuous third-order sliding mode controller for a robot system is given in Theorem 3.

**Theorem 3.** Consider uncertain robot manipulators with a dynamic model described by equation (6).
Figure 4. Structure of the fuzzy quasi-continuous third-order sliding mode controller. QC3C: quasi-continuous third-order sliding mode controller; SOSM: second-order sliding mode.

Figure 5. 3-DOF PUMA560 robot. 3-DOF: three-degrees-of-freedom.
Figure 6. Tracking performance of the QC2C and FQC2C: without uncertainties: (a) Link 1, (b) Link 2 and (c) Link 3. QC2C: quasi-continuous second-order sliding mode controller; FQC2C: fuzzy quasi-continuous second-order sliding mode controller.
The second-order exact differentiator in equation (33) is assumed to converge to the real sliding manifold derivative in finite time. The QC3C law given by equation (31) ensures that the sliding surface \( s \) asymptotically converges to zero in a finite time.

**Proof.** Similar to the proof of stability of QC2C in Theorem 2 in ‘Design of FQC2C’ section, we let the Lyapunov function be \( V = \frac{1}{2} s^T s \). Differentiating \( V \) with respect to time gives

\[
\dot{V} = s^T \dot{s} = s^T \left[ M^{-1}(x_l)u_{QC3} + \Delta \right]
\]

\[
= s^T \left[ -\frac{\dot{s} + 2(|s| + |s|^{2/3})^{-1/2}(\dot{s} + |s|^{2/3} \text{sign}(s))}{|\dot{s}| + 2(|s| + |s|^{2/3})^{1/2}} \right] + \Delta
\]

\[
= s^T \Delta - s^T k \text{sign}(s)
\]

\[
\dot{s} = \frac{\Delta \text{sign}(s) + 2 \text{sign}(s) \left( \text{sign}(s) + |s|^{2/3} \right)}{|\dot{s}| + 2 \left( |s| + |s|^{2/3} \right)^{1/2}}
\]

because

\[
\frac{\text{sign}(s) + |s|^{2/3}}{|\dot{s}| + 2 \left( |s| + |s|^{2/3} \right)^{1/2}} \leq 1
\]

Consequently

\[
\frac{\text{sign}(s) + 2 \text{sign}(s) \left( \text{sign}(s) + |s|^{2/3} \right)}{|\dot{s}| + 2 \left( |s| + |s|^{2/3} \right)^{1/2}} \leq 1
\]

Therefore, from equation (36) if we select \( k \geq \bar{k} \), then \( V < 0 \) holds. It means that the sliding surface \( s \) asymptotically converges to zero in a finite time.

**Figure 7.** Control effort of the QC2C: without uncertainties: (a) input 1, (b) input 2 and (c) input 3.

QC2C: quasi-continuous second-order sliding mode controller.
In order to eliminate the chattering of the QC3C, an adaptive fuzzy controller is employed. The FQC3C illustrated in Figure 4 is designed as

\[ u = u_{eq} + u_{FQC3} \tag{37} \]

Here

\[ u_{FQC3} = M(x_1) \hat{k} \left( \frac{z_2 + 2(|z_1| + |z_0|^{2/3})^{-1/2} \times (|z_1| + |z_0|^{2/3} \text{sign}(z_0))}{|z_2| + 2(|z_1| + |z_0|^{2/3})^{1/2}} \right) \tag{38} \]

where \( \hat{k} = u_f \) is the output of the adaptive fuzzy control which is defined same as in ‘Fuzzy quasi-continuous second-order sliding mode controller’ section.

**Remark 1.** No matter what the control input is, the observer (super-twisting SOSM observer, robust exact differentiator) has a finite time convergence; therefore, the closed loop observer-controller can be successfully applied without any stability problem.

**Remark 2.** To further eliminate chattering, a small scalar \( \rho \) is added to the formula of the QC2C and FQC2C \( \tag{39} \)

\[ u_{QC2} = -M(x_1) \hat{k} \left( \frac{z_1 + |z_0|^{1/2} \text{sign}(z_0)}{|z_1| + |z_0|^{1/2} + \rho} \right) \]

and

\[ u_{FQC2} = -M(x_1) u_f \left( \frac{z_1 + |z_0|^{1/2} \text{sign}(z_0)}{|z_1| + |z_0|^{1/2} + \rho} \right) \tag{40} \]

However, there is a tradeoff between chattering and the robustness property. If \( \rho \) is large, the chattering is smaller but robustness is reduced, and vice versa.

**Remark 3.** The property in Remark 2 is also true when it is applied to the QC3C and FQC3C.

Figure 8. Control effort of the FQC2C: without uncertainties: (a) input 1, (b) input 2 and (c) input 3. FQC2C: fuzzy quasi-continuous second-order sliding mode controller.
In this case, smoothed QC3C and FQC3C are obtained as

\[
u_{QC3} = M(x_1)k \left( \frac{z_2 + 2(\lvert z_1 \rvert + \lvert z_0 \rvert^{2/3})^{-1/2}}{\lvert z_2 \rvert + 2(\lvert z_1 \rvert + \lvert z_0 \rvert^{2/3})^{1/2} + \rho} \right)
\]

and

\[
u_{FQC3} = M(x_1)u_f \left( \frac{z_2 + 2(\lvert z_1 \rvert + \lvert z_0 \rvert^{2/3})^{-1/2}}{\lvert z_2 \rvert + 2(\lvert z_1 \rvert + \lvert z_0 \rvert^{2/3})^{1/2} + \rho} \right)
\]

Figure 9. Tracking performance of the QC2C and FQC2C: with uncertainties: (a) Link 1, (b) Link 2 and (c) Link 3. QC2C: quasi-continuous second-order sliding mode controller; FQC2C: fuzzy quasi-continuous second-order sliding mode controller.

**Simulation results**

In order to verify the effectiveness of the proposed fuzzy quasi-continuous HOSM controllers, its overall procedure is simulated for a PUMA560 robot in which the first three joints are used. The PUMA560 robot is a well-known industrial robot that has been widely used in industrial applications and robotic research. The explicit dynamic model and parameter values necessary to control the robot are given by Armstrong et al. The three-degrees-of-freedom (3-DOF) PUMA560 robot is considered with the last three joints locked. A kinematic description of the robot is given in Figure 5. In this simulation, the parameters of the PUMA560 robot are taken from Armstrong et al. These parameters are used to build the nominal dynamic model. As the robot dynamic equation described in equation (1), \( f(q, \dot{q}) \) is a function of the positions and velocities that denotes unmodeled
Matlab/Simulink is used to perform all simulations with the sampling time set at $10^{-3}$s. The object of this article is to design a controller using QC2C, FQC2C, QC3C and FQC3C and find out the best scheme to track the desired trajectories as time goes to infinity. Here, the desired joint trajectories to be tracked are $q_d = [q_{1d}, q_{2d}, q_{3d}]^T$ with $q_{1d} = \cos(\frac{t}{2\pi}) - 1$, $q_{2d} = \cos(\frac{t}{2\pi} + \frac{\pi}{2})$ and $q_{3d} = \sin(\frac{t}{2\pi} + \frac{\pi}{2}) - 1$. The design parameters of the QC2C and QC3C are chosen to be larger than the upper bound of the assumed uncertainties $k = 35$. The parameters of the super-twisting SOSM observer are chosen as $\alpha_1 = 5$ and $\alpha_2 = 6$. First- and second-order differentiator parameters are all set to $\lambda = 5$. For smoothing of QC2C and QC3C, the positive scalar in equation (39)–(41) is chosen as $\rho = 0.1$. These parameters were chosen by running simulations with various values and selecting a suitable one to achieve the best transient control performance. In this simulation, we consider the robotic system in two operational situations: without uncertainties ($f(q, \dot{q}) = 0$) and with assumed uncertainties.

**Without uncertainties**

The tracking performance of QC2C and FQC2C is shown in Figure 6. From this figure, we see that FQC2C converges faster and with less error than...
 QC2C. The control efforts of QC2C and FQC2C are shown in Figures 7 and 8, respectively. Significant chattering is generated by QC2C when the system remains in the sliding manifold $s = \dot{s} = 0$ (Figure 7). By applying the adaptive fuzzy system, FQC2C almost eliminates the chattering compared to the conventional QC2C (Figure 8).

**With uncertainties**

The simulation set is divided into two parts. For the first simulation set, we verify the capability of FQC2C compared with that of a QC2C. In the second simulation set, the tracking performance of the FQC3C is compared with those of QC3C.

For the first set of simulations, performances of QC2C and FQC2C are shown in Figures 9 to 12. Figure 9 shows that both the QC2C and FQC2C have good motion tracking. The transient response of the FQC2C is faster than that of the QC2C. The control efforts of QC2C and FQC2C are shown in Figures 10 and 11, respectively. Due to the large sliding gain values selected to confront the uncertainties that exist in the robot system, the QC2C produced significant chattering. The control efforts of the QC2C are shown in Figure 10. By using the fuzzy system to adjust the sliding mode gain, the chattering and control effort is much reduced by FQC2C as shown in Figure 11. The estimated uncertainty upper bound generated by fuzzy output of the FQC2C is shown in Figure 12.

In the second set of the simulation, the proposed FQC3C is compared with the QC3C. The results are shown in Figures 13 to 16. In Figure 13, which shows the tracking performance, the transient response of the QC3C is too slow. This is the major drawback of QC3C compared to the QC2C. By employing an adaptive fuzzy system, the transient response of
FQC3C is much improved. On the other hand, when comparing Figure 14 with Figure 15, the chattering that exists in the control effort of FQC3C is smaller than that with QC3C. The fuzzy outputs of FQC3C can be used to adjust the sliding mode gain, as shown in Figure 16.

In order to evaluate the performance of the quasi-continuous HOSM controllers, QC3C and FQC3C are compared with those of QC2C and FQC2C, respectively. As shown in Figures 9 and 13, the tracking response of the QC2C is much faster than that of the QC3C (e.g. the convergence time of QC2C of Link 1 is approximately \( t = 1.64 \text{ s} \), while QC3C is approximately \( t = 5.82 \text{ s} \)). This is also true when we compared the FQC2C (Figures 9 and 12) with that of FQC3C (Figures 13 and 16). The tracking response of the FQC2C is faster (e.g. the convergence time of FQC2C of Link 1 is approximately \( t = 0.67 \text{ s} \), while FQC3C is approximately \( t = 0.98 \text{ s} \)). In contrast, the QC3C and FQC3C produce less tracking error and smoother control torques. Comparison of Figure 14 with Figure 10 and Figure 15 with Figure 11 shows that the control efforts of the QC3C and FQC3C are smoother than those of the QC2C and FQC2C, respectively. It shows that QC3C, FQC3C has less tracking error than QC2C, FQC2C, respectively. Comparison in root mean square error of the uncertain robot system under the QC2C, FQC2C, QC3C and FQC3C (when the system is convergent: at the time 6–10 s) are shown in Table 2.

In conclusion, QC3C provides more accurate motion tracking and smooth control action than QC2C. Unfortunately, the time response is much slower compared with that of QC2C. This problem can be solved by employing an adaptive fuzzy system. The FQC3C has less tracking error, smooth
Figure 13. Tracking performance of the QC3C and FQC3C: with uncertainties: (a) link 1, (b) link 2 and (c) link 3. QC3C: quasi-continuous third-order sliding mode; FQC3C: fuzzy quasi-continuous third-order sliding mode.

Figure 14. Control effort of the QC3C: with uncertainties: (a) input 1, (b) input 2 and (c) input 3. QC3C: quasi-continuous third-order sliding mode.
Figure 14. Continued.

Figure 15. Control effort of the FQC3C: with uncertainties: (a) input 1, (b) input 2 and (c) input 3. FQC3C: fuzzy quasi-continuous third-order sliding mode.
control action and a fast transient response. From the results, FQC3C could be a good selection for robot controller design.

Remark 3. Due to the properties of QC3C and FQC3C, QC2C and FQC2C for the robotic system in the presence of uncertainties are also true when it is applied for the system without uncertainties. Only simulation results for QC2C and FQC2C are considered for the robotic system without uncertainties to reduce the length of this article.

Conclusion

In this article, an output feedback tracking control scheme based on quasi-continuous HOSM, robust exact differentiators and fuzzy technique has been investigated for uncertain robot manipulators. A quasi-continuous HOSM-based exact differentiator provides less chattering and higher accuracy compared with conventional sliding mode and other HOSM controllers were designed. In addition, a fuzzy inference mechanism was employed to adaptively tune the sliding mode gain to improve tracking response, tracking accuracy and smooth control action when the system remains on the sliding manifold. The results of

![Figure 16. Estimated upper bound of the uncertainties of FQC3C: with uncertainties: (a) fuzzy output 1, (b) fuzzy output 2 and (c) fuzzy output 3.](image)

FQC3C: fuzzy quasi-continuous third-order sliding mode.

Table 2. Comparison in root mean square error of the QC2C, FQC2C, QC3C and FQC3C for robot system with uncertainties.

<table>
<thead>
<tr>
<th></th>
<th>Link 1</th>
<th>Link 2</th>
<th>Link 3</th>
</tr>
</thead>
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<tr>
<td>QC2C</td>
<td>6.6439 × 10^{-4}</td>
<td>3.1433 × 10^{-4}</td>
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<tr>
<td>FQC2C</td>
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<td>1.8297 × 10^{-4}</td>
<td>2.8172 × 10^{-4}</td>
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<tr>
<td>QC3S</td>
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<td>5.1507 × 10^{-5}</td>
<td>1.9606 × 10^{-4}</td>
</tr>
<tr>
<td>FQC3C</td>
<td>1.4083 × 10^{-5}</td>
<td>4.4992 × 10^{-5}</td>
<td>1.5038 × 10^{-4}</td>
</tr>
</tbody>
</table>

QC2C: quasi-continuous second-order sliding mode controller; FQC2C: fuzzy quasi-continuous second-order sliding mode controller; QC3C: quasi-continuous third-order sliding mode controller; FQC3C: fuzzy quasi-continuous third-order sliding mode controller.

QC2C: quasi-continuous second-order sliding mode controller; FQC2C: fuzzy quasi-continuous second-order sliding mode controller; QC3C: quasi-continuous third-order sliding mode controller; FQC3C: fuzzy quasi-continuous third-order sliding mode controller.
computer simulations for a 3-DOF PUMA560 robot, comparing the QC2C, FQC2C, QC3C and FQC3C, verify the effectiveness of the proposed strategy.

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**Conflict of interest**
None declared.

**References**